Chapter VIII
Priority Arguments

In this chapter some standard results of classical recursion theory are lifted to every \( \Sigma_1 \) admissible ordinal \( \alpha \) by techniques hinted at in earlier chapters. Since the classical proofs use the \( \Sigma_2 \) admissibility of \( L(\omega) \), the proofs to come may be regarded as instances of the austere art of making \( \Sigma_1 \) admissibility do the work of \( \Sigma_2 \). The initial technique relies strongly on the \( L \)-ness of \( L \), the ability of \( L \) to support downward Skolem–Löwenheim arguments. The later technique depends on combinatorial consequences of \( \Sigma_1 \) admissibility. Its dynamic nature makes it applicable to \( \Sigma_1 \) admissible structures for which hull-collapsing arguments fail.

1. \( \alpha \)-Finite Injury via \( \alpha^* \)

In this section and the next it will be shown that there exist \( \alpha \)-recursively enumerable sets \( A \) and \( B \) such that neither is \( \alpha \)-recursive in the other. The method extends that applied in Section 5.VII, to construct a hyperregular, non-\( \alpha \)-recursive, \( \alpha \)-recursively enumerable set. The injury sets become more complex. Back in Chapter VII each negative requirement was injured at most once for the sake of each positive requirement of higher priority. Such simplicity is rare in the present chapter. Consequently the \( \alpha \)-recursive projection of \( \alpha \) into \( \alpha^* \), which arranges that each negative requirement be opposed by less than-\( \alpha^* \) positive requirements of higher priority, does not always compel the injury sets to be \( \alpha \)-finite. In Section 2 below, when \( \alpha^* = \alpha \) and there is a greatest \( \alpha \)-cardinal, it will be necessary to project \( \alpha \) downward by means of a carefully chosen \( \Sigma_2 \) function.

1.1 Strategy. Define \( \{ p^{-1}_\epsilon \}_B \) for each \( \epsilon < \alpha^* \) as in the beginning of the proof of Theorem 5.5.VII. Thus \( A \leq_{wA} B \) iff \( A = \{ p^{-1}_\epsilon \}_B \) for some \( \epsilon < \alpha^* \). The requirements on \( A \) and \( B \) are as follows.

Requirement 2\( \epsilon \): If \( \{ p^{-1}_\epsilon \}_B \) is a total function, then \( A \neq \{ p^{-1}_\epsilon \}_B \).

Requirement 2\( \epsilon + 1 \): Same as req 2\( \epsilon \) with \( A \) and \( B \) interchanged.

Let \( \{ Z_\epsilon | \epsilon < \alpha^* \} \) be a collection of simultaneously \( \alpha \)-recursive, pairwise disjoint, unbounded subsets of \( \alpha \). The strategy for satisfying req 2\( \epsilon \) consists of finding an \( x \in Z_\epsilon \) such that:

\[
\text{if } \{ p^{-1}_\epsilon \}_B(\epsilon) = 0, \text{ then } x \in A.
\]
(Recall that $A(x)$, the characteristic function of $A$, is 1 when $x \in A$.) As in the proof of Theorem 5.5.VII, $\{p^{-1}\epsilon\}^B(x)$ is approximated at stage $\sigma$ of the construction of $A$ and $B$ by $\{p^{-1}\epsilon\}^B_{<\sigma}(x)$, an $\alpha$-recursive function of $\epsilon$, $\sigma$ and $x$. $B^\prec \sigma$ is that part of $B$ enumerated prior to stage $\sigma$.

Suppose $\{p^{-1}\epsilon\}^B_{<\sigma}(x) = 0$ and $x \notin A^\prec \sigma$ for some $x \in Z_\epsilon$. Then at stage $\sigma$ an attempt to satisfy req $2\epsilon$ can be made by adding $x$ to $A$ and making a commitment to preserve the computation of $\{p^{-1}\epsilon\}^B_{<\sigma}(x)$ through all future stages. The commitment reduces to: exclude all the elements of some $\alpha$-finite $K$ from $B$, where $K$ is the set of negative facts about $B^\prec \sigma$ used in the computation of $\{p^{-1}\epsilon\}^B_{<\sigma}(x)$. If the attempt is made at stage $\sigma$, and if the resulting commitment is honored at all future stages, then $x \in A$, $\{p^{-1}\epsilon\}^B(x) = 0$ and req $2\epsilon$ is satisfied.

The commitment is broken the first time some member of $K$ is added to $B$ for the sake of some req $2\epsilon_0 + 1$ at some stage $\tau > \sigma$. In that event req $2\epsilon$ is said to be injured at stage $\tau$ for the sake of req $2\epsilon_0 + 1$. Let

$$I_{2\epsilon} = \{\tau | \text{req } 2\epsilon \text{ is injured at stage } \tau\}.$$ 

$I_{2\epsilon}$ is called an injury set. Define $I_{2\epsilon+1}$ similarly. The injury sets are simultaneously $\alpha$-recursively enumerable. Once it is seen that all injury sets are $\alpha$-finite, it is not difficult to show all requirements are met.

Two devices are employed to limit the extent of injury sets. The first assigns priorities. Requirement $x$ has higher priority than requirement $y$ if $x < y$. The priority method consists of allowing requirement $y$ to be injured for the sake of requirement $x$ only if $x$ has higher priority than $y$. Thus req $2\epsilon$ can be injured by adding some element to $B$ for the sake of req $2\epsilon_0 + 1$ only when $2\epsilon_0 + 1 < 2\epsilon$. The second device enhances the first. It indexes the requirements so as to form the shortest possible list compatible with a $\Sigma^0_1$ construction. The shorter the list, the smaller the set of requirements of higher priority than a given requirement. In this section the requirements are indexed by ordinals less than $\alpha^*$ with the aid of $p$, an $\alpha$-recursive projection of $\alpha$ into $\alpha^*$. In the next section they are indexed by ordinals less than a certain ordinal below $\alpha^*$ with the aid of a $\Sigma^0_2$ projection.

1.2 Construction of $A$ and $B$. At each stage attention is paid to only one requirement, but every requirement is paid attention unboundedly often. Let $\sigma$ be a stage at which attention is paid to req $2\epsilon$.

First suppose an attempt was made to satisfy req $2\epsilon$ prior to stage $\sigma$ and no subsequent injury was inflicted on req $2\epsilon$ prior to stage $\sigma$. To elaborate: there was a stage $\sigma_0 < \sigma$ at which some element of $Z_\epsilon$ was put in $A$ and a commitment was made to keep some $\alpha$-finite $K$ from touching $B$; that commitment has been honored up to now, in short, $K \subseteq cB^\prec \sigma$. There is clearly no need to make a new attempt at stage $\sigma$ to satisfy req $2\epsilon$. Let $A^\sigma = A^\prec \sigma$ and $B^\sigma = B^\prec \sigma$.

Now suppose every attempt to satisfy req $2\epsilon$ made prior to stage $\sigma$ was injured prior to stage $\sigma$. Let $m_\epsilon(\sigma, \epsilon)$ be the supremum of all $y$ such that prior to stage $\sigma$, a commitment was made to keep $y$ out of $A$ for the sake of req $2\epsilon_0 + 1$ for some $\epsilon_0 < \epsilon$. Then any $y > m_\epsilon(\sigma, \epsilon)$ can be added to $A$ at stage $\sigma$ without injuring any
requirement of higher priority than req $2\varepsilon$. Define 
\[ w(\sigma, \varepsilon) = \mu y [y > m_B(\sigma, \varepsilon) \quad \& \quad y \in Z_\varepsilon - A^{<\sigma}]. \]

(1) If $\{p^{-1}q^\varepsilon\}^{<\sigma}(w(\sigma, \varepsilon)) = 0$, then 
\[ A^{\sigma} = A^{<\sigma} \cup \{w(\sigma, \varepsilon)\} \quad \text{and} \quad B^{\sigma} = B^{<\sigma}; \]
in addition a commitment is made to keep all of $K$ out of $B$, where $K$ is an $\alpha$-finite set of negative membership facts about $B^{<\sigma}$ needed for the computation of (1). If (1) is false, then $A^\sigma = A^{<\sigma}$ and $B^\sigma = B^{<\sigma}$.

Req $2\varepsilon + 1$ is handled similarly. End of construction.

The next lemma captures the so-called $\alpha$-finite injury method. The analogous Friedberg–Muchnik lemma says that the $n$-th requirement is injured less than $2^n$ times.

1.3 Lemma. Suppose $\beta$ is an infinite $\alpha$-cardinal and $\delta < \beta$. Then injury set $I_\delta$ is $\alpha$-finite and has $\alpha$-cardinality $< \beta$.

Proof. Let $\beta_0 \leq \beta$ be a regular $\alpha$-cardinal such that $\delta < \beta_0$. For the sake of an induction, assume for $\gamma < \delta$,

$I_\gamma$ is $\alpha$-finite and has $\alpha$-cardinality $< \beta_0$.

the $I_\gamma$'s are simultaneously $\alpha$-recursively enumerable, and so by Lemma 2.3.VII,

$\cup \{I_\gamma | \gamma < \delta\}$ is $\alpha$-finite and has $\alpha$-cardinality $< \beta_0$.

Now define 
\[ A_\gamma = \{\sigma | \text{an attempt is made at stage } \sigma \text{ to satisfy req. } \gamma\}. \]

Observe that $A_\gamma$ and $I_\gamma$ are interlaced: between any two elements of either set lies a member of the other. Hence if either set is $\alpha$-finite, then so is the other, and in addition their $\alpha$-cardinalities differ by at most 1. Consequently 

(1) $\cup \{A_\gamma | \gamma < \delta\}$ is $\alpha$-finite and has $\alpha$-cardinality $< \beta_0$.

$I_\delta$ is a subset of $\cup \{A_\gamma | \gamma < \delta\}$, and so can be viewed as an $\alpha$-recursively enumerable subset of some ordinal less than $\beta_0$. Since $\beta_0 \leq \alpha^*$, $I_\delta$ is $\alpha$-finite. □

Theorem 1.4 solves half of Post's problem. The other half is in the next section.

1.4 Theorem. Suppose $\alpha^* < \alpha$, or $\alpha^* = \alpha$ and there is no greatest $\alpha$-cardinal. Then there exist $\alpha$-recursively enumerable sets $A$ and $B$ such that $A \triangleleft_{wa} B$ and $B \triangleleft_{wa} A$. 
Proof. Let \( A \) and \( B \) be the sets enumerated in subsection 1.2. Assume \( \{p \varepsilon^{-1}\}^B \) is total with the intent of showing \( A \neq \{p \varepsilon^{-1}\}^B \). According to formula (1) of the proof of Lemma 1.3, there exists a \( \sigma_0 \) such that for \( \delta \leq 2 \varepsilon \), all attempts to satisfy \( \text{req} \ 2 \varepsilon \) occur prior to stage \( \sigma_0 \).

Case 1: there was an attempt at some stage \( \sigma \) to satisfy \( \text{req} \ 2 \varepsilon \), and the associated commitment was honored at all subsequent stages. Then

\[
\{p^{-1} \varepsilon\}^B_{\sigma}(w(\sigma, \varepsilon)) = 1 = \{p^{-1} \varepsilon\}^B(w(\sigma, \varepsilon))
\]

and \( A^\sigma(w(\sigma, \varepsilon)) = 0 = A(w(\sigma, \varepsilon)) \).

Case 2: otherwise. Let \( \sigma \geq \sigma_1 \geq \sigma_0 \) and suppose \( \sigma \) and \( \sigma_1 \) are stages at which attention is paid to \( \text{req} \ 2 \varepsilon \). Then

\[
m_B(\sigma, \varepsilon) = m_B(\sigma_1, \varepsilon) \quad \text{and} \quad w_B(\sigma, \varepsilon) = w_B(\sigma_1, \varepsilon) \notin A^< \sigma.
\]

No attempt is made to satisfy \( \text{req} \ 2 \varepsilon \) at stage \( \sigma \), so

\[
\{p^{-1} \varepsilon\}^B_{\sigma}(w_B(\sigma, \varepsilon)) \neq 1.
\]

Since \( \{p^{-1} \varepsilon\}^B \) is total, it follows that

\[
\{p^{-1} \varepsilon\}^B(w_B(\sigma_1, \varepsilon)) \neq 1.
\]

Thus \( w_B(\sigma_1, \varepsilon) \) is a witness to the inequality of \( A \) and \( \{p^{-1} \varepsilon\}^B \). \( \square \)

The technology of the next section is more than is needed to solve Post's problem. Exercise 1.5 describes a much less technical solution. Sections 1 and 2 are intended to introduce ideas needed for further results such as the uniform solution of subsection 3.7.

1.5 Exercise. Assume \( \alpha^* = \alpha \). Construct \( A \) and \( B \) as in subsection 1.2 with one additional proviso: suppose \( \text{req} \ x \) is injured at stage \( \sigma \) for the sake of \( \text{req} \ y \); then \( \text{req} \ x \) can be injured at stage \( \tau > \sigma \) for the sake of \( \text{req} \ z \) only if \( z < y \). Show each injury set is finite. Show \( A \) and \( B \) are \( \alpha \)-recursively enumerable, \( A \not\leq_{wa} B \), and \( B \not\leq_{wa} A \).

2. \( \alpha \)-Finite Injury and Tameness

In this section it is assumed that \( \alpha^* = \alpha \) and there exists a greatest \( \alpha \)-cardinal, call it \( \mathbb{N} \). The solution to Post's problem given in the previous section now fails, because \( \alpha^* \) is no longer the limit of regular \( \alpha \)-cardinals, and consequently Lemma 1.3 is not applicable to all \( \delta < \alpha^* \). One way around the difficulty is to project \( \alpha \) into
something potentially smaller than $\alpha^*$. The new mode of projection will be $\Sigma^2_\alpha$ rather than $\Sigma^1_\alpha$, and the range of the projection will be some ordinal multiple of $\aleph_0$. Lemma 1.3 will apply to each block of requirements of length $\aleph_0$, and a further idea will be involved to control the injuries within a short union of blocks. The arguments of this section rely strongly on the fine structure of $L$ and are as concrete as possible. Lemma 2.7 provides rare bounds on $\alpha$-finite injury sets. In succeeding sections the method is dealt with more abstractly so as to be applicable to a wide range of $\Sigma_1$ admissible sets.

2.1 Tame $\Sigma^2_\alpha$ Maps. The notion of tameness was invented by M. Lerman to clarify the proof of Theorem 2.6. It has many applications. Let $f: \alpha \to \alpha$ be $\Sigma^2_\alpha$. $f$ has an $\alpha$-recursive approximation that will not surprise students of classical recursion theory. Since $f \in \Sigma^2_\alpha$, there is a $D \in \Delta^0_\alpha$ such that

$$f(x) = y \leftrightarrow L(\alpha) \models (\text{Ev})(u)D(u, v, x, y)$$

for all $x, y < \alpha$. Define $\alpha$-recursive $w$ and $g$ by:

(1) \hspace{1cm} w(\sigma, x) = \mu u < \sigma (u)_\alpha D(u, (w)_0, x, (w)_1); \\
(2) \hspace{1cm} g(\sigma, x) = (w(\sigma, x))_1.$

2.2 Proposition. $f \in \Sigma^2_\alpha$ iff there exists an $\alpha$-recursive $g$ such that $f(x) = \lim_{\sigma} g(\sigma, x)$.

Proof. Suppose $f(x) = \lim_{\sigma} g(\sigma, x)$. Then

$$f(x) = y \leftrightarrow (E\sigma)(\tau)_{\tau \geq \sigma} [g(\sigma, x) = y].$$

Now suppose $f$ is $\Sigma^2_\alpha$. Define $w(\sigma, x)$ and $g(\sigma, x)$ as in (1)–(2) of subsection 2.1. Fix $x$ and let $w$ be the least $\langle v, y \rangle$ such that $(u)D(u, v, x, y)$ holds in $L(\alpha)$. Then $(w)_1 = f(x)$. Define

$$h(z) \approx \mu u \sim D(u, (z)_0, x, (z)_1).$$

$h$ is partial $\alpha$-recursive and defined for all $z < w$. Let

$$\tau = \sup\{h(z) | z < w\}.$$ 

$\tau < \alpha$ thanks to the $\Sigma_1$ admissibility of $L(\alpha)$. For all $\sigma > \tau$, $w(\sigma, x) = w$ and $g(\sigma, x) = f(x)$. □

Let $f: \delta \to \alpha$ for some $\delta \leq \alpha$. $f$ is said to be tame $\Sigma^2_\alpha$ if there exists an $\alpha$-recursive $g$ such that

$$(\gamma)_{\gamma < \delta} (E\tau)(\sigma)_{\sigma > \tau}(x)_{x < \gamma} [g(\sigma, x) = f(x)].$$
Thus $\lim_{\sigma, x} g(\sigma, x) = f(x)$, and so $f \in \Sigma^2_\omega$ as in the first part of the proof of Proposition 2.2. The tameness of $f$ refers to the way $g$ approximates $f$ on proper initial segments of the domain of $f$. Let $g_\sigma$ denote $\lambda x [g(\sigma, x)]$. Then

$$\forall \gamma < \delta (E\tau)(\sigma)_{\sigma > \tau} [g_\sigma \upharpoonright \gamma = f \upharpoonright \gamma].$$

A $\Sigma^2_\omega$ function need not be tame $\Sigma^2_\omega$ (cf. Exercise 2.14).

The tame $\Sigma^2_\omega$ projectum of $\alpha$, denoted by $t_\alpha (\alpha)$, is

$$\mu(y) [f \in \text{tame } \Sigma^2_\omega \& f \text{ maps } \gamma \text{ onto } \alpha].$$

**Warning:** it can happen that $\alpha > t_\alpha (\alpha)$ and $t_\alpha (\alpha)$ is not an $\alpha$-cardinal.

### 2.3 Lemma (Lerman).

Let $g(\sigma, x)$ be an $\alpha$-recursive function such that $\lim_{\sigma} g(\sigma, x)$ exists for all $x$. Suppose $\lim_{\sigma} g(\sigma, x)$ maps $\alpha$ one-one into $\delta$. Assume: $(\forall \gamma < \delta) (E\tau)(y)_{\gamma < \gamma}$ either

$$(\exists x) [g(\sigma, x) \neq y] \quad \text{or} \quad (\exists x) [g(\sigma, x) = y].$$

Then $t_\alpha (\alpha) \leq \delta$.

**Proof.** The map $\lim_{\sigma} g(\sigma, x)$ is collapsed and inverted. Let $h(\sigma, y)$ be the least member of

$$\{g(\sigma, w) | w \leq \sigma \} - \{h(\sigma, z) | z < y \}$$

if there is one, and zero otherwise. Let $\delta_0$ be the ordertype of the range of $\lambda x [\lim_{\sigma} g(\sigma, x)]$. Then for each $y < \delta_0$, $\lim_{\sigma} h(\sigma, y)$ exists and is equal to $y$-th smallest member of the range of $\lambda x [\lim_{\sigma} g(\sigma, x)]$. For $y < \delta_0$, define

$$k(\sigma, y) = \mu x_{\leq \sigma} [g(\sigma, x) = h(\sigma, y)].$$

Then $k(\sigma, y)$ is $\alpha$-recursive, and $\lim_{\sigma} k(\sigma, y)$ is a tame $\Sigma^2_\omega$ map of $\delta_0$ onto $\alpha$.

### 2.4 Corollary. $\alpha^* \leq \alpha^*$.

**Proof.** Let $g: \alpha \to \alpha^*$ be one-one, into, and $\alpha$-recursive. Define $g(\sigma, x) = g(x)$ for all $\sigma, x < \alpha$. Then $g(\sigma, x)$ satisfies the hypotheses of Lemma 2.3 ($\delta = \alpha^*$) with the aid of Proposition 2.1.VII.
The $\Sigma_2^a$ cofinality of $\alpha$, denoted by $\sigma2cf(\alpha)$, is

$$\mu(\text{Eff}[f \in \Sigma_2^a & f \text{ is strictly increasing}
$$
$$& \text{ dom } f = \beta & \text{ sup range } f = \alpha].$$

$\sigma2cf(\alpha)$ measures the failure of $L(\alpha)$ to be $\Sigma_2$ admissible. Thus $L(\alpha)$ is $\Sigma_2$ admissible iff $\sigma2cf(\alpha) = \alpha$.

The greatest cardinal of $\alpha$, denoted by $gc(\alpha)$, is the greatest $\alpha$-cardinal, if there is one, and $\alpha$ otherwise.

2.5 Lemma. If $t\sigma2p(\alpha) > gc(\alpha)$, then

$$t\sigma2p(\alpha) = gc(\alpha) \cdot \sigma2cf(\alpha).$$

Proof. Let $t\sigma2p(\alpha) = gc(\alpha) \cdot \gamma + \lambda$, where $0 < \gamma \leq \alpha$ and $\lambda < gc(\alpha)$; and let $f$ be a tame $\Sigma_2^a$ map from $t\sigma2p(\alpha)$ onto $\alpha$. Suppose $\lambda > 0$. Then $f[gc(\alpha) \cdot \gamma]$ is $\alpha$-finite by the tameness of $f$. Let $t$ be a one-one, $\alpha$-recursive map of $\alpha - f[gc(\alpha) \cdot \gamma]$ onto $\alpha$. Then $f_0$, defined by

$$f_0(x) = t(f(gc(\alpha) \cdot \gamma + x),$$

is a tame $\Sigma_2^a$ map from $\lambda$ onto $\alpha$. But that is impossible because $\lambda < t\sigma2p(\alpha)$.

Thus $t\sigma2p(\alpha) = gc(\alpha) \cdot \gamma$. Suppose $\gamma$ is not a limit ordinal. Then $t\sigma2p(\alpha) = gc(\alpha) \cdot (\gamma - 1) + gc(\alpha)$, and the argument of the previous paragraph can be repeated with $gc(\alpha)$ in place of $\lambda$ to obtain a tame $\Sigma_2^a$ map from $gc(\alpha)$ onto $\alpha$. An impossibility since $gc(\alpha) < t\sigma2p(\alpha)$.

Thus $t\sigma2p(\alpha) = gc(\alpha) \cdot \gamma$ for some limit $\gamma$. For each $\delta < \gamma$, let

$$h(\delta) = \sup \{ f(x) | x < gc(\alpha) \cdot \delta \} + \delta.$$
Post's problem for $\Sigma_1$ admissible ordinals. It relies on a tame $\Sigma_2$ phenomenon derived from stability properties in $L$.

2.6 Theorem (Sacks & Simpson 1972). There exist two $\alpha$-recursively enumerable sets such that neither is weakly $\alpha$-recursive in the other (Post's problem).

Proof. By Theorem 1.4 it is safe to assume $\alpha^* = \alpha$ and there is a greatest $\alpha$-cardinal, call it $\aleph$. By Lemma 2.6.VII, $\alpha$ is the limit of $\alpha$-stable ordinals. Let

\[ 0 = \delta_0 < \aleph < \delta_1 < \delta_2 < \cdots < \delta_y < \cdots (\gamma < \lambda_0) \]

be a listing of all $\alpha$-stable ordinals beyond $\aleph$. A tame $\Sigma^2_{\alpha}$ map $f$ from $\aleph \cdot \lambda_0$ onto $\alpha$ will be defined in a moment. (It is possible that $\tau \neg \exists p(x) < \aleph \cdot \lambda_0$.) The $\alpha$-stable ordinals divide $\alpha$ into $\lambda_0$ blocks. Block $\gamma$ is $[\delta_y, \delta_{\gamma+1})$. Let $h_\gamma$ be the least $\alpha$-finite map of $\aleph$ onto block $\gamma$. Define

\[ f(x) = h_\gamma(x - (\aleph \cdot \gamma)) \quad \text{if} \quad \aleph \cdot \gamma \leq x < \aleph \cdot (\gamma + 1). \]

$f$ maps $\aleph \cdot \lambda_0$ onto $\alpha$.

Guessing tamely at $f$ proceeds as follows. At stage $\sigma$ let

\[ \aleph = \delta_1^\sigma < \delta_2^\sigma < \cdots < \delta_y^\sigma < \cdots (\gamma < \lambda_0^\sigma) \]

be a listing of all $\sigma$-stable ordinals beyond $\aleph$. ($\beta$ is $\sigma$-stable if $\beta < \sigma$ and $L(\beta) <_1 L(\sigma)$.) If $\beta$ is $\sigma$-stable and $\sigma \geq$ some $\alpha$-stable ordinal $\geq \beta$, then $\beta$ is $\alpha$-stable. Hence

\[ \delta_y^\sigma = \delta_y \quad \text{for all} \quad \sigma \geq \delta_{\gamma+1}. \]

Thus $\delta_\gamma$ is a tame $\Sigma^2_{\alpha}$ function of $\gamma$.

Let $h_\gamma^\sigma$ be the least $\alpha$-finite map of $\aleph$ onto $[\delta_y^\sigma, \delta_{\gamma+1})$ in $L(\sigma)$, if there is one; otherwise $h_\gamma^\sigma = 0$. Then $h_\gamma^\sigma = h_\gamma$ for all $\sigma \geq \delta_{\gamma+2}$. Let

\[ g(\sigma, x) = h_\gamma^\sigma(x - (\aleph \cdot \gamma)) \quad \text{if} \quad \aleph \cdot \gamma \leq x < (\gamma + 1) \]

for all $x < \aleph \cdot \lambda_0^\sigma$. Then $f$ is tame $\Sigma^2_{\alpha}$ via the $\alpha$-recursive approximation $g$. In essence $f$ is tame because a correct guess of one $\alpha$-stable ordinal implies a correct guess of all lesser $\alpha$-stable ordinals.

The construction of $A$ and $B$ proceeds as in subsections 1.1-1.2 save that requirements are indexed by ordinals less than $\aleph \cdot \lambda_0$. For each $\epsilon < \aleph \cdot \lambda_0$, req $2\epsilon$ is: if \{f $\epsilon$\}$^B$ is total, then $A \neq \{f \epsilon\}^B$. req $2\epsilon + 1$ is similar with $A$ and $B$ exchanged. Stage $\sigma$ unfolds as it did in subsection 1.2 with \{p $^{-1}$ \epsilon\}$^B$ replaced by \{g(\sigma, \epsilon)\}$^B$. Consequently there is a new reason for making repeated attempts to satisfy req $2\epsilon$. Reduction procedure \{g(\sigma, \epsilon)\} varies with $\sigma$. The resulting disturbance dies down.
quickly. By the $\Sigma^2$ tameness of $f$ via $g$,
\[
(x)_{x < \lambda_0} (ET)(\sigma)^{\varepsilon > \varepsilon}[g(\sigma, \varepsilon) = f(\varepsilon)].
\]

For $\varepsilon < \lambda_0$ and $i < 2$, define injury set
\[
I'_{\varepsilon} = \{\sigma | \text{req}(2\varepsilon + i) \text{ is injured at stage } \sigma \}.
\]

The next lemma should be compared with Lemma 1.3.

2.7 Lemma. $I^i_{\delta_{y+2}} \in L(\delta_{y+2})$ ($y < \lambda_0$ & $\beta < \lambda^1$).

Proof. Req$(2\varepsilon + i)$ is said to be active at stage $\sigma$ if it is injured at stage $\sigma$, or if an attempt to satisfy it is made at stage $\sigma$, or if
\[
(\tau)^{\tau < \sigma}(E\eta)^{\eta < \sigma}[g(\eta, \varepsilon) \neq g(\sigma, \varepsilon)].
\]

By induction on $\gamma$ the following is proved: all activity occasional by req $2(\lambda \cdot \gamma + \beta) + i$ takes place before stage $\tau$ for some $\tau < \delta_{y+2}$.

Req$(2\varepsilon + i)$ is said to be in block $y$ if
\[
\lambda \cdot y \leq \varepsilon < \lambda \cdot (y + 1).
\]

Fix $\gamma$. By induction all activity with respect to requirements in block $y$, for all $y < \gamma$, takes place before stage $\delta_{y+1}$. $h_y \in L(\delta_{y+2})$, so there is a $\tau_0 < \delta_{y+2}$ such that
\[
(\sigma)^{\sigma \geq \tau_0}[g(\sigma, \varepsilon) = f(\varepsilon)].
\]

Now an induction on $\beta$ shows that req $z(\lambda \cdot \gamma + \beta) + i$ is inactive after stage $\tau$ for some $\tau < \delta_{y+2}$. The induction on $\beta$ proceeds in the same fashion as the induction on $\delta$ in the proof of Lemma 1.3. After stage $\tau_0$, injuries to, and attempts to satisfy, req $2(\lambda \cdot \gamma + \beta) + i$ are interlaced. As in the proof 1.3, Lemma 2.3.VII is applied to show $I^i_{\lambda \cdot \gamma + \beta}$ is $\alpha$-finite and has $\alpha$-cardinality less than $\lambda$. In short, the argument of 1.3 works within block $\beta$. Let
\[
\tau_0 \leq \sigma_0 < \sigma_1 < \ldots < \sigma_i < \ldots (i < \rho)
\]
be a listing of $I^i_{\lambda \cdot \gamma + \beta} - \tau_0$. $\{\sigma_i | i < \rho\}$ is an $\alpha$-recursively enumerable set defined by a $\Sigma^1_1$ formula whose parameters are $i$, $\gamma$, $\beta$, $\tau_0$ and $\lambda$. ($\lambda$ is the only parameter needed for the enumeration of $A$ and $B$.) Since $\rho < \lambda < \delta_{y+2}$, the $\alpha$-stability of $\delta_{y+2}$ pins down the $\sigma_i$'s. Suppose $\{\sigma_i | i < j\} \subseteq L(\delta_{y+2})$. Proceed by induction on $j$. If $j \leq \rho$, then $\{\sigma_i | i < j\} \subseteq L(\delta_{y+2})$, because $\delta_{y+2}$ is $\Sigma^1_1$ admissible. Hence $\{\sigma_i | i < \rho\} \subseteq L(\delta_{y+2})$.

The proof of Theorem 2.6 is completed as in the proof of Theorem 1.4. \qed
2.8-2.15 Exercises

2.8. Assume \( A, B \subseteq \alpha \). \( L[B, \alpha] \) was defined in subsection 3.5.VII. \( L[B, \alpha]^+ \) is the least \( C \supseteq L[B, \alpha] \) such that \( \langle C, B \rangle \) is \( \Sigma_1 \) admissible. \( A \) is said to be \( \alpha \)-computable from \( B \) (in symbols \( A \leq \alpha B \)) iff \( A \) is \( \Delta_1 \) over \( \langle L[B, \alpha]^+, B \rangle \). Suppose \( B \) is \( \alpha \)-recursively enumerable and hyperregular. Show

\[
A \leq \alpha B \iff A \leq \alpha B.
\]

2.9. Show there exist \( \alpha \)-recursively enumerable sets \( A \) and \( B \) such that \( A \not\leq \alpha B \) and \( B \not\leq \alpha A \).

2.10. Show that \( \sigma \text{cf}(\alpha) \) equals

\[
\mu\beta(E_f)[f \in \Sigma_2^\alpha & \text{dom } f = \beta & \sup \text{range } f = \alpha].
\]

(In other words the clause "\( f \) is strictly increasing" can be dropped from the definition of \( \sigma \text{cf}(\alpha) \).)

2.11. Show that \( \tau \sigma \text{p}(\alpha) \) equals

\[
\mu\gamma(E_f)[f \in \text{tame } \Sigma_2^\alpha & f: \gamma \overset{1-1}{\text{onto}} \alpha].
\]

2.12. Find an \( \alpha \) such that \( \tau \sigma \text{p}(\alpha) < \alpha \) and \( \tau \sigma \text{p}(\alpha) \) is not an \( \alpha \)-cardinal.

2.13. Find an \( \alpha \) such that \( \tau \sigma \text{p}(\alpha) < \aleph_\cdot \lambda_0 \), where \( \aleph \) and \( \lambda_0 \) are as in the proof of Theorem 2.6.

2.14. Show \( \alpha \) is \( \Sigma_2 \) admissible iff every \( \Sigma_2^\alpha \) function is tame \( \Sigma_2^\alpha \).

2.15. Suppose \( f \in \Sigma_2^\alpha \) and \( \text{dom } f \leq \sigma \text{cf}(\alpha) \). Show \( f \) is tame \( \Sigma_2^\alpha \).

3. Dynamic Versus Fine-Structure

An argument in higher recursion theory, particularly an argument about "recursively enumerable" sets, is said to be fine-structure in character if it relies on the collapsing (or condensation) method associated with \( L \). The proof of Theorem 2.6 is such an argument, because the stable ordinals \( \delta_\cdot \gamma (\gamma < \lambda_0) \) owe their existence to Lemma 2.6.VII, whose proof makes explicit use of Mostowski's collapsing map. A dynamic argument relies on combinatoric reasoning about cofinalities and projecta. It may form hulls, but it does not collapse them. The regular sets theorem, 4.2.VII, is proved dynamically. A less obvious example is Theorem 5.3.VII, the existence of a non-\( \alpha \)-recursive, hyperregular set. It makes use of Lemma 2.3.VII and Proposition 2.1.VII, both proved dynamically. Both 2.3 and 2.1 have a dependence on admissibility. If \( \alpha \) is not \( \Sigma_1 \) admissible, then 2.3 can fail, but 2.1 remains true by a fine-structure argument.
Dynamic methods possess the force needed to operate outside $L$ and are in harmony with classical recursion theory. Fine structure techniques are more delicate. They keep careful count of injuries to requirements. Shore's density theorem, Chapter IX, is a powerful combination of both approaches.

The purpose of this section is to enlarge the repertoire of dynamic methods.

3.1 Lerman's Tame $\Sigma_2$ Approach. The fine-structure aspects of the proof of Theorem 2.6 (Post's problem) can be eliminated systematically as follows. (The proof of 2.6 employed a tame $\Sigma_2^*$ map from $\mathbb{N} \cdot \lambda_\alpha$ onto $\alpha$ when $\alpha^* = \alpha$ and $\mathbb{N} = gc(\alpha) < \alpha$.) Requirements are indexed by ordinals less than $t\sigma_2p(x)$.

The treatment of requirements breaks into two cases.

(i) $t\sigma_2p(\alpha) \leq gc(\alpha)(\leq \alpha)$.
(ii) $t\sigma_2p(\alpha) = gc(\alpha) \cdot \sigma_2cf(\alpha)$.

By Lemma 2.5 either (i) or (ii) holds for all $\alpha$. If (i) holds, then $\varepsilon$ is less than some $\alpha$-cardinal. If (ii) holds, then $\varepsilon$ sits in block $\gamma$, that is,

$$gc(\alpha) \cdot \gamma < \varepsilon < gc(\alpha) \cdot (\gamma + 1),$$

and can be viewed as less than some $\alpha$-cardinal modulo $gc(\alpha)$.

The reasoning behind Lemma 1.3 applies to both cases. Fix $\varepsilon$ and choose $\tau$ so that the $\alpha$-recursive approximation of $f \uparrow (\varepsilon + 1)$ is correct from stage $\tau$ onward. Assume $I_x$, the $x$-th injury set, is $\alpha$-finite for all $x < \varepsilon$. If (i) holds, then $I_\varepsilon$ is $\alpha$-finite, as in 1.3, by the combinatoric lemma (2.3.VII). Note the use of Proposition 2.1.VII in the proof of 1.3.

Suppose (ii) holds. Let

$$J_\delta = \cup \{I_x | gc(\alpha) \cdot \delta \leq x < gc(\alpha) \cdot (\delta + 1)\}.$$

Assume $J_\delta$ is $\alpha$-finite for all $\delta < \gamma$. Since $J_\delta$ is a $\Sigma_2^*$ function of $\delta$, and $\gamma < \sigma_2cf(\alpha)$, it follows that $\cup \{J_\delta | \delta < \gamma\}$ is $\alpha$-finite. Hence there is a stage $\sigma_\gamma(\geq \tau)$ after which all activity with respect to requirements in block $\delta$, for all $\delta < \gamma$, ceases. Then requirements in $[gc(\alpha) \cdot \gamma, \varepsilon)$ can be handled the same way as requirements in $[0, \varepsilon)$ were handled in case (i). In short: $\gamma < \sigma_2cf(\alpha)$ implied a bound on activity in the first $\gamma$ blocks; and the combinatoric lemma implies a bound on activity up to any point within a block, since the length of a block is an $\alpha$-cardinal.

It still has to be shown that $J_\gamma$ is $\alpha$-finite. Let $I'_\gamma$ be the set of stages after $\sigma_\gamma$ at which req $x$ is injured. It suffices to show

(1) $\cup \{I'_x | gc(\alpha) \cdot \gamma \leq x < gc(\alpha) \cdot (\gamma + 1)\}$

is $\alpha$-finite. The combinatoric lemma implies (by induction on $x$ in block $\gamma$) that each $I'_x$ is $\alpha$-finite and has $\alpha$-cardinality less than $gc(\alpha)$. The simultaneous enumeration of the $I'_x$'s gives rise to a partial $\alpha$-recursive, 1-1 map of (1) into $gc(\alpha) \cdot gc(\alpha)$. If $b$ is the $\beta$-th member of $I'_\gamma$ to be enumerated, then $b$ is mapped to $<\beta, x - gc(\alpha)>$. Suppose (1) is not $\alpha$-finite. Then there is an $\alpha$-recursive, one-one map from $\alpha$ into (1), and
from (1) into $g_0(\alpha)$. Hence $\alpha^* < g_0(\alpha)$, and so $t\sigma^2p(\alpha) \leq g_0(\alpha)$ by Corollary 2.4. But that contradicts the hypothesis of case (ii). □

3.2 Shore's Blocking Method. The notion of block is carried a step further. For Shore there is no conflict between requirements in the same block. For example, the requirements in a given block might all be of the form $A \neq \{\varepsilon\}^B$. The success of the method turns on a lemma concerning cofinalities. Define $\sigma_2\text{cf}^*(\delta)$, the $\Sigma^+_2$ cofinality of $\delta$, to be

$$
\mu \beta(\text{Eh})[h \in \Sigma^+_2 \text{ & } h \text{ is strictly increasing & dom } h = \beta \text{ & sup range } h = \delta].
$$

Thus $\sigma_2\text{cf}^*(\alpha) = \sigma_2\text{cf}(\alpha)$.

3.3 Lemma (Shore). $\sigma_2\text{cf}^*(\alpha^*) = \sigma_2\text{cf}(\alpha)$.

Proof. Let $f : \alpha \to \alpha^*$ be $\alpha$-recursive, one-one and into. Suppose $g$ is a strictly increasing $\Sigma^+_2$ map from $\beta$ into $\alpha^*$ with range unbounded in $\alpha^*$. Then there exists a strictly increasing $\Sigma^+_2$ map $h$ from $\beta$ into $\alpha$ with unbounded range. A rough approximation of $h$ is $f^{-1}g$. The details are as follows. Define

$$
h(x) = \sup\{f^{-1}(y) | y < g(x) \text{ & } y \in \text{range } f\} + x.
$$

The "+x" insures that $h$ is strictly increasing. Proposition 2.1.VII implies $h$ is $\Sigma^+_2$; for each $x < \beta$ there exists $\alpha$-finite sets $w_1$ and $w_2$ such that

$$
w_1 \cup w_2 = g(x) \text{ & } w_1 \subseteq \text{range } f \text{ & } w_2 \cap \text{range } f = \phi.
$$

Thus $h(x) = z$ iff

(1) $$(Ew)(Ew_1)(Ew_2)[w = g(x) \text{ & } w = w_1 \cup w_2 \text{ & } w \subseteq \text{range } f \text{ & } w_2 \cap \text{range } f = \phi \text{ & } z = \sup\{f^{-1}(y) | y \in \text{range } w_1\} + x].
$$

(1) is easily seen to be $\Sigma^+_2$ save for perhaps one detail. The formula, $w \subseteq \text{rg } f$, rendered as

(2) $$
(u)_{u \in w}(Ew)(Ex)(f(x) = u),
$$

appears to be $\Pi^+_2$, but is in fact $\Sigma^+_1$. Let $P(x, u, v)$ be a $\Delta^0_\alpha$ formula such that

$$
f(x) = u \leftrightarrow (Ev)P(x, u, v).
$$

Then (2) is equivalent to

$$(Eq)(u)_{u \in w}(Ex)(Ev)(P(x, u, v)).$$
Thus the $\Sigma_1$ admissibility of $L(\alpha)$ is needed to show $h$ is $\Sigma^2_2$. Hence $\sigma 2 cf (\alpha) \leq \sigma 2 cf (\alpha^*)$.

Now suppose a strictly increasing $\Sigma^2_2$ map $h$ from $\sigma 2 cf (\alpha)$ into $\alpha$ is given. A good approximation of the desired cofinality map $g$ from $\sigma 2 cf (\alpha)$ into $\alpha^*$ is $fh$. Clearly $fh$ is $\Sigma^2_2$. The range of $fh$ is unbounded in $\alpha^*$; otherwise range $fh \subseteq \delta$ for some $\delta < \alpha^*$, and then range $h$ is bounded by $\sup f^{-1}[\delta] < \alpha$. The only difficulty is that $fh$ may not be strictly increasing. Define

$$\beta = \mu \gamma(\epsilon_t)[t \in \Sigma^2_2 \& \ dom t = \gamma \& \ sup range t = \alpha].$$

Let $t$ be a $\Sigma^2_2$ map from $\beta$ into $\alpha$ with unbounded range. $ft$ is $\Sigma^2_2$ and its range is unbounded in $\alpha^*$. Define $\beta_0$ to be

$$\mu \gamma(\epsilon_p)[p \in \Sigma^2_2 \& \ dom p = \gamma \& \ sup range = \alpha^*].$$

then $\beta_0 \leq \beta \leq \sigma 2 cf (\alpha)$. Let $g_0$ be a $\Sigma^2_2$ map from $\beta_0$ into $\alpha^*$ with range unbounded in $\alpha^*$. To show $\sigma 2 cf (\alpha^*) \leq \sigma 2 cf (\alpha)$, it suffices to transform $g_0$ into a strictly increasing function $g$ with the same domain ($= \beta_0$).

For $x < \beta_0$, let

$$g(x) = \sup \{g_0(y)|y \leq x\} + x.$$ 

To see $g$ is $\Sigma^2_2$, let $k(x, \sigma)$ be an $\alpha$-recursive function such that

$$g_0(x) = \lim_{\sigma} k(\sigma, x) \quad (x < \beta_0).$$

$k$ exists by Proposition 2.2. Let

$$m(x) = \mu \sigma (\tau)_{\tau \geq \sigma}[k(\sigma, x) = k(\tau, x)].$$

$m$ is $\Sigma^2_2$, since $m(x) = \sigma$ iff

$$\tau \geq \sigma[k(\sigma, x) = k(\tau, x)] \quad \& \quad (\rho)^{\rho < \sigma}(E \gamma)^{\gamma < \sigma}[k(\rho, x) \neq k(\gamma, x)].$$

Hence for each $x < \beta_0 \leq \beta$,

$$\{m(y)|y \leq x\}$$

is bounded below $\alpha$. In other words, for each $x < \beta_0$

$$\{g_0(y)|y \leq x\}$$

is $\Sigma^2_2$ and serves as the principal part of a $\Sigma^2_2$ definition of $g$. Note that $\sup \{g_0(y)|y \leq x\}$ equals $\sup \{k(\sigma, y)|y \leq x\}$ for all sufficiently large $\sigma$. □
Note well the use of $\Sigma_1$ admissibility in the above proof to show a formula is $\Sigma_2^*$.

3.4 Blocking. As in subsection 1.1 the requirements for Post’s problem are indexed by ordinals less than $\alpha^*$. Let $g$ be a strictly increasing, $\Sigma_2^*$ map from $\sigma2cf(\alpha^*)$ into $\alpha^*$ with range unbounded in $\alpha^*$. Shore’s blocking method uses $g$ to distribute requirements as follows. $p$ is an $\alpha$-recursive, 1-1 map of $\alpha$ into $\alpha^*$.

- **Block** $2\delta$: all requirements of the form $A \neq \{p^{-1}\varepsilon\}^p$, where $g(\delta) \leq \varepsilon < g(\delta + 1)$.
- **Block** $2\delta + 1$: all requirements of the form $B \neq \{p^{-1}\varepsilon\}^A$, $g(\delta) \leq \varepsilon < g(\delta + 1)$.

The procedure for trying to meet requirements is the one established in subsection 1.1. Hence there is no conflict between requirements in the same Shore block. Block $\gamma$ is said to have a higher priority than block $\rho$ if $\gamma < \rho$. Thus a requirement in block $2\delta$ can be injured only by an attempt to satisfy a requirement in block $2\eta + 1$ for some $\eta < \delta$. In order to proceed effectively at stage $\sigma$, the $\Sigma_2^*$ cofinality map $g(\delta)$ is replaced by an $\alpha$-recursive approximation $g(\sigma, \delta)$ supplied by Proposition 2.2.

Define $\lim_{\tau<\sigma} g(\tau, x)$ to be $z$ iff

$$(E\gamma)_{\gamma < \sigma}(\tau)_{\gamma < \sigma} [g(\tau, x) = z].$$

For each $\rho < \sigma2cf^*(\alpha^*)$, let $J_\rho$ be the set of all $\sigma$ such that:

1. (i) an attempt is made to satisfy, or an injury occurs to, a requirement in $\cup \{\text{block } x | x \leq \rho\}$ at stage $\sigma$; or
2. (ii) $g(\sigma, x) \neq \lim_{\tau<\sigma} g(\tau, x)$ for some $x \leq \rho$.

As in Section 1 it suffices to see $J_\rho$ is $\alpha$-finite.

Assume $J_x$ is $\alpha$-finite for $x < \rho$. Then $J_x$ is a $\Sigma_2^*$ function of $x$ below $\rho$. Since $\rho < \sigma2cf^*(\alpha^*)$, Lemma 3.3 implies

$$\sup \{J_x | x < \rho\} = \sigma_0 < \alpha.$$ 

Choose $\sigma_1 > \sigma_0$ so that $(\tau)_{\tau \geq \sigma_1} [g(\tau, \rho) = g(\sigma_1, \rho)]$. Let $J_\rho^1 = J_\rho - \sigma_1$. Suppose $\sigma \in J_\rho^1$. The only activity at stage $\sigma$ with respect to block $\delta$ is an attempt to meet one requirement in that block. (Recall that the procedure for meeting requirements established in subsection 1.1 allows an attempt on at most one requirement at each stage.) Each requirement in block $\delta$ will be attempted at at most one stage in $J_\rho^1$. Thus $J_\rho^1$ is in $\alpha$-recursive, 1-1 correspondence with an $\alpha$-recursively enumerable subset of block $\rho$. That subset is $\alpha$-finite, since block $\rho$ is shorter than $\alpha^*$. Hence $J_\rho^1$ is $\alpha$-finite.

3.5 Post’s Problem for Admissible Sets. Let $A$ be a $\Sigma_1$ admissible set as defined in Section 1.VII. Recall that $B \subseteq A$ is said to be $A$-recursively enumerable if $B$ is $\Sigma_1^A$ and $A$-recursive if $A$ is $\Delta_1^A$. Also $z$ is said to be $A$-finite if $z \in A$. Many, but not all, of
the notions of \( \alpha \)-recursion theory extend to \( A \). For example, regularity can only mean: \( B \subseteq A \) is regular iff \((B \cap z) \subseteq A \) for all \( z \in A \). On the other hand, the notion of \( \Sigma_1 \) projectum does not seem to make sense for all \( \Sigma_1 \) admissible \( A \)'s. If \( A \) is \((Ad(2^\omega))\), the least \( \Sigma_1 \) admissible set with \( 2^\omega \) as an element, then there does not appear to be any useful way of defining the \( \Sigma_1 \) projectum of \( A \).

To formulate Post's problem for all \( A \), all that is needed is an extension of "\( \alpha \)-recursive in" to "\( A \)-recursive in". Suppose \( B, C \subseteq A \). \( B \) is \( A \)-recursive in \( C \), symbolically \( B \leq_A C \), if there exist partial \( \alpha \)-recursive functions \( \phi \) and \( \psi \) such that

\[
x \leq B \leftrightarrow (Ez)(Ey)[\phi(x, y, z) = 0 \& y \subseteq C \& z \subseteq A - C],
\]

\[
x \leq A - B \leftrightarrow (Ez)(Ey)[\psi(x, y, z) = 0 \& y \subseteq C \& z \subseteq A - C],
\]

for all \( x \in A \). (\( y \) and \( z \) range over \( A \).) Clearly \( \leq_A \) is reflexive and transitive. Two sets have the same \( A \)-degree if each is \( A \)-recursive in the other. The notion of "weakly \( A \)-recursive in" \( (\leq_{wA}) \) is defined by substituting "\( x \in A \)" for "\( x \subseteq A \)" in the definition of \( \leq_A \).

One formulation of Post's problem for \( A \) is: do there exist \( A \)-recursively enumerable sets \( B \) and \( C \) such that \( B \leq_{wA} C \) and \( C \leq_{wA} B \)? Another formulation that asks for more: \( B \leq_{wA} C \) and \( C \leq_{wA} B \)? Still another asks for less: does there exist an \( A \)-recursively enumerable set \( B \) such that \( B \) is neither \( A \)-recursive nor complete? (\( B \) is complete if every \( A \)-recursively enumerable set is \( A \)-recursive in \( B \).

Simpson showed that \( HC \), the set of all hereditarily countable sets, yields a negative answer to the third formulation above, if the axiom of determinateness is assumed. Later Harrington showed, as a theorem of \( ZF \), that there exists a countable \( \Sigma_1 \) admissible set for which the third formulation has a negative answer. His proof can be found in Chong 1984.

On the other hand, Simpson 1974b and Stoltenberg–Hansen 1977 showed that the second (and strongest) formulation has a positive answer if \( A \) can be suitably well-ordered. The condition to be considered here is: \( A \) is effectively well-orderable, that is, there exists a one-one \( A \)-recursive map of \( A \) onto \( \text{ord}(A) \), the least ordinal not in \( A \). If \( A \) is effectively well-orderable, then the notion of \( \Sigma_1 \) projectum is meaningful for \( A \), and the dynamic solutions of Post's problem given above via Lerman's tame \( \Sigma_2 \) projectum (subsection 3.1) or Shore's blocking method (subsection 3.2) apply to \( A \). The fine structure approach (Theorem 2.6) fails in \( A \), because it is based on collapsing arguments that need initial segments of \( L \) to succeed. The next proposition is an aid to understanding why dynamic methods succeed in \( A \) when \( A \) is effectively well-orderable.

### 3.6 Proposition
Assume \( A \) is a \( \Sigma_1 \) admissible set. Let \( \alpha = \text{ord}(A) \). Then (i) iff (ii).

(i) \( A \) is effectively well-orderable.
(ii) There exists a \( \Delta_1^A \) \( B \subseteq \alpha \) such that \( A = L[B, \alpha] \).

**Proof.** \( L[B, \alpha] \) was defined in subsection 3.5.VII. If (ii) holds, then the natural enumeration of \( L(\alpha) \), described in subsection 1.7.VII, extends to one of \( L[B, \alpha] \), and yields an effective well-ordering of \( A \).
Suppose (i) holds. Let $f: A \to \alpha$ be a one-one, onto, $\Sigma^A$ map. $B$ will encode all sets in $A$ by relations on ordinals. Let $x \in A$, and define $tc(x)$, the transitive closure of $x$, to be the least transitive $y \supseteq x$. (Note that $tc(x)$, as a function of $x$, is $\Sigma^A$.) Define

$$\delta \in f^{-1}(\gamma) \text{ by } f^{-1}((\gamma).$$

then $\langle tc(x), e \rangle \cong \langle f[tc(x)], e_f \rangle$. Let

$$r = \langle f[x], f[tc(x)], e_f \cap (f[tc(x)])^2 \rangle.$$

$r$ is an effective code for $x$ in that the passage from $r$ to $x$ can be accomplished inside $L(\alpha, r)$. Note that the collection of all such $r$ is $\alpha$-recursive. $r$ can be further coded as a set of ordinals.

Thus it will be enough for $B$ to encode all sets of ordinals in $A$. $\alpha$-recursive functions $g, h: \alpha \to \alpha$ are defined by recursion. Let $f_0$ be a one-one, $\alpha$-recursive map of $\alpha$ onto $\alpha^2$. Define

$$h(\delta) = (\sup_{\gamma < \delta} h(\gamma)) + \sup_{\gamma < \delta} f_0(\gamma),$$

$$g(h(\delta) + x) = \begin{cases} 0 & \text{if } x \in f_0(\delta) \\ 1 & \text{if } x \notin f_0(\delta) \end{cases} \quad (x < \sup_{\gamma < \delta} f_0(\delta)).$$

Each $y \in \alpha^2$ can be recovered from a sufficiently long initial segment of $h$ and $g$. Since $h, g \subseteq \alpha^2$, they can be encoded by a $B \subseteq \alpha$. 

**3.7 Theorem.** Let $A$ be an effectively well-orderable, $\Sigma^1_1$ admissible set. Then there exist two $\alpha$-recursively enumerable sets such that neither is weakly $\alpha$-recursive in the other.

**Proof.** By Shore's blocking method. Define

$$\alpha^*_\alpha = \mu \gamma (Eg \[ g \in \Sigma^A_1 \& \ g: \alpha \to \gamma \]_1).$$

$\alpha^*_\alpha$ is the $\Sigma^1_1$ projection of $A$. The proof of Proposition 2.1.VII was entirely dynamic in nature and so applies to $\alpha^*_\alpha$. Suppose $X \subseteq \delta \subseteq \alpha^*_\alpha$ and $X$ is $\alpha$-recursively enumerable. Let $f$ be a one-one, $\alpha$-recursive map of $A$ onto $\text{ord}(A)$. With the aid of $f$, $X$ becomes the range of a partial, one-one, $\alpha$-recursive map $g$ with domain equal to an initial segment of $\text{ord}(A)$. The domain of $g$ cannot be $\text{ord}(A)$ because $\delta \subseteq \alpha^*_\alpha$. Thus the domain of $g$, and hence $X$, is $\alpha$-finite.

Similarly the proof of Proposition 2.2 shows that each $\Sigma^2_1$ function from $\text{ord}(A)$ into $\text{ord}(A)$ is the limit of an $\alpha$-recursive approximation. And the proof of Lemma 3.3 becomes a proof of

$$\sigma 2cf^A(\alpha^*_\alpha) = \sigma 2cf^A(\text{ord}(A)).$$
The arguments of subsection 3.4 transfer to $A$ straightforwardly. The two "incomparable" sets obtained are subsets of $\alpha$. □

3.8 Uniform Solutions to Post's Problem. Call a $\Sigma^1_2$ subset of $L(\alpha)$ lightface if its $\Sigma^1_2$ definition has all its parameters in $\omega$, and boldface otherwise. Thus the set of ordinals less than $\alpha^*$ in $L(\alpha)$ is boldface, but not lightface, $\Sigma^1_2$, because its definition needs $\alpha^*$ as a parameter. All the solutions of Post's problem given prior to this section yield "incomparable" sets that are boldface $\Sigma^1_2$. The parameters needed arise out of various definable projecta and confinalities. For example, the blocking argument of subsection 3.4 needs the parameters occurring in the $\Sigma^1_2$ definition of $p$ and the $\Sigma^2_2$ definition of $g$. The solution given below avoids all such parameters and is uniform in $\alpha$.

3.8 Theorem (R. Shore 1974). There exist integers $m$ and $n$ such that for all $\alpha$: the $m$-th and $n$-th lightface $\Sigma^x_1$ sets have the property that neither is weakly $\alpha$-recursive in the other.

Proof. Curiously the uniform construction, viewed locally, is similar to that given in Section 1 when $\alpha^* < \alpha$. The idea is to divide $\alpha$ into intervals that are independent in the sense that requirements in different intervals do not conflict. The top of each interval is to be a $\Sigma^1_2$ admissible ordinal with strictly smaller $\Sigma^1_2$ projectum. The $\alpha$-stable ordinals are $\Sigma^1_2$ admissible and, according to Lemma 2.7, define somewhat independent intervals.

Let $\{t(\sigma, \delta) | 0 < \delta < \sigma_0\}$ be a list in ascending order of the $\sigma$-stable ordinals. Set $t(\sigma, 0) = \omega$ and $t(\sigma, \sigma_0) = \sigma$.

Define $t_\delta = t(\alpha, \delta)$. Thus $t_\delta$ is the $\delta$-th $\alpha$-stable ordinal. $t(\sigma, \delta) = t_\delta$ if $\sigma \geq t_\delta$, and $t(\sigma, \delta)$ is a lightface $\alpha$-recursive function of $\sigma$ and $\delta$. Hence $t_\delta$ is a lightface, tame $\Sigma^2_2$ function of $\delta$.

Let $h$ be the universal, partial $\alpha$-recursive function central to the proof of Lemma 2.6.VII. Recall that $h$ is lightface $\Sigma^1_2$, uniformly in $\alpha$, and that $h[y]$ is a $\Sigma^1_2$ substructure, and an initial segment, of $L(\alpha)$ for every infinite $y < \alpha$. It follows that

\[
(0) \quad \text{(i)} \ h[t_\delta + 1] = L(t_\delta + 1) \quad \text{and} \quad \text{(ii)} \ t^*_\delta + 1 \leq t_\delta
\]

for all $\delta < \sigma_0$. (ii) is obtained from (i) by inverting $h$.

Since $t^*_\delta + 1 < t_{\delta + 1} (\delta < \sigma_0)$, the local strategy of the uniform construction can be modeled on that of subsection 1.1. Suppose

\[
t_\delta \leq \rho < \sigma < t_{\delta + 1}.
\]

At stage $\sigma$ the intention is to consider only requirements of the form $A \neq \{\rho\}^B$ (req $2\rho$) or $B \neq \{\rho\}^A$ (req $2\rho + 1$). The priorities are governed by $f_\sigma$, a partial, one-one $\Sigma^1_{\delta + 1}$ map from $t^*_\delta + 1$ onto $t_{\delta + 1}$ defined below. The local strategy yields the desired global result: req $\rho$ is satisfied prior to stage $t_{\delta + 1}$, and that will be so thanks to the $\alpha$-stability of $t_{\delta + 1}$. Of course at stage $\sigma$ the local strategy can only be guessed.
The guesses must converge properly, and above all, must be lightface \(\alpha\)-recursive uniformly in \(\alpha\).

Define \(f_\delta = h g_\delta\), where \(g_\delta\) is the first member of \(L(t_{\delta+1})\) to be a one-one map of \(t_\delta^*\) onto \(t_\delta\). \(h\upharpoonright t_\delta\) is \(\Sigma_1^{\text{I+1}}\) because \(t_{\delta+1}\) is stable. Thus \(f_\delta\) is a partial \(\Sigma_1^{\text{I+1}}\) function from \(t_{\delta+1}^*\) onto \(t_{\delta+1}\). It is safe to assume \(f_\delta\) is one-one (cf. Exercise 1.16.VII). The only non-trivial parameter in the \(\Sigma_1\) definition of \(f_\delta\) is \(g_\delta\). \(g_\delta\) is the best guess in \(L(\sigma)\) at a one-one map of the \(\sigma\)-cardinality of \(t(\sigma,\delta)\) onto \(t(\sigma,\delta)\). (Recall that for any \(\Sigma_1\) admissible \(\gamma\), if \(\gamma^* < \gamma\), then \(\gamma^*\) is the greatest \(\gamma\)-cardinal.) As \(\sigma\) approaches \(t_{\delta+1}\)

\[
t(\sigma, \delta) = t_\delta, \\
\sigma\text{-card}(t_\delta) = t_{\delta+1}^*,
\]

and \(g_\sigma^\sigma = g_\delta\). Let \(h^\sigma\) be the result of restricting the \(\Sigma_1^\sigma\) definition of \(h\) to \(L(\sigma)\). Then \(h^\sigma\) is lightface \(\Sigma_1^\sigma\).

Define \(f_\delta^\sigma = h^\sigma g_\delta^\sigma\). Then

\[
\lim_{\sigma \to t_{\delta+1}} f_\delta^\sigma \upharpoonright z = f_\delta^\sigma \upharpoonright z.
\]

Suppose \(\omega \leq \rho < \sigma\). Let \(t_{m(\sigma, \rho)}\) be the greatest \(\sigma\)-stable ordinal \(\leq \rho\). Then

\[
\lim_{\sigma \to t_{\delta+1}} t_{m(\sigma, \rho)} = t_\delta.
\]

(Remember that \(\lim_{\sigma \to y} x_\sigma = x\) means \((E \tau)_{\tau < y} (\sigma)_{\tau \leq \sigma < y} (x_\sigma = x)\).)

The uniform solution.

Stage \(\sigma < \omega\). Identical with the Friedberg–Muchnik solution to Post's problem.

Stage \(\sigma \geq \omega\). Suppose \(\rho < \sigma\) and an attempt was made at stage \(\tau < \sigma\) to satisfy \(\text{req} \rho\). That attempt, specifically the negative requirement imposed by that attempt on \(A\) or \(B\), is now discarded if

\[
t_{m(\tau, \rho)} > t_{m(\sigma, \rho)}.
\]

If \(r < \sigma\), then \(r\) belongs to a unique block of the form \([t(\sigma, \delta), t(\sigma, \delta + 1))\). The local priority assigned to \(\text{req} \ r\) at stage \(\sigma\) is \((f_\delta^\sigma)^{-1} (r)\). As usual \(\text{req} \ x\) has higher priority than \(\text{req} \ y\) if the priority assigned to \(x\) is less than that assigned to \(y\).

\(\text{Req} \ r\) needs attention at stage \(\sigma\) if:

(i) every attempt to satisfy \(\text{req} \ r\) prior to stage \(\sigma\) was injured prior to stage \(\sigma\), or discarded prior to or at stage \(\sigma\); and

(ii) there is an opportunity to satisfy \(\text{req} \ r\) at stage \(\sigma\) that does not threaten injury to any \(\text{req}\) of higher priority in the same block as \(r\), or to any requirement in any block below that of \(r\).

Go to the lowest block in which there is a requirement that needs attention. In that block go to the highest priority requirement that needs attention, and take the least opportunity to satisfy it.
3. Dynamic Versus Fine-Structure

End of uniform solution.

Fix $\delta < \alpha_0$ and $z < t_{\delta + 1}$ in order to show by induction that $J^\delta_x$ (the set of all $\sigma$ such that an injury to, or attempt to satisfy, $\text{req}_f(z)$ occurs at stage $\sigma$) is $t_{\delta + 1}$-finite. As in (1) and (2) there is a $\sigma_1 \in [t_{\delta}, t_{\delta + 1})$ such that for all $\sigma \geq \sigma_1$:

$$t(\sigma, \delta) = t_\delta,$$

$$f_\delta(z + 1) = f_\delta^\sigma(z + 1)$$

$$t(\sigma, \delta + 1) > \sup f_\delta[z + 1].$$

Then for all $x \leq z$ at stage $\sigma \geq \sigma_1$:

(3) no prior attempt to satisfy $\text{req}_f(x)$ is discarded;

(4) $\text{req}_f(x)$ is injured only for the sake of $\text{req}_f(y)(y < x)$.

(4) follows from the induction hypothesis that $J^\gamma_x \in L(t_{\gamma + 1})$; for all $\gamma < \delta$ and $v < t_{\gamma + 1}$.

By induction the $J^\delta_x$'s ($x < z$) are simultaneously $t_{\delta + 1}$-recursively enumerable, and each is $t_{\delta + 1}$-finite. The combinatoric lemma (2.3.VII) implies $\cup \{J^\delta_x \mid x < z\}$ is $t_{\delta + 1}$-finite. After stage $\sigma_1$ and after $U \{J^\delta_x \mid x < z\}$ is enumerated, there is at most one attempt to satisfy $\text{req}_f$. Hence $J^\delta_x$ is $\alpha$-finite.

Fix $r \in [t_{\delta}, t_{\delta + 1})$ to see $\text{req}_f$ is met. $r = f_\delta(z)$ for some $z < t_{\delta + 1}$. Assume $\text{req}_f$ is of the form $A \neq \{r\}^B$. Choose $\sigma_2 \in (\sigma_1, t_{\delta + 1})$ so that

$$\cup \{J^\delta_x \mid x \leq z\} \subseteq \sigma_2.$$  

$\sigma_1$ is as above. Any attempt to satisfy $\text{req}_f$ involves a witness $w$ and the inequation $A(w) \neq \{r\}^B(w)$. $w$ is drawn from a witness set $Z^\delta_x$. The witness sets are pairwise disjoint, unbounded in $x$, and simultaneously lightface $\alpha$-recursive. The $\alpha$-stability of $t_{\delta + 1}$ implies that $Z^\delta_x - t_\delta$ is unbounded in $t_{\delta + 1}$. Let $w_0$ be a member of $Z^\delta_x - t_\delta$ not put in $A$ prior to stage $\sigma_2$. Then $w_0$ is never put in $A$, i.e. $A(w_0) = 0$. Suppose $\{r\}^B(w_0) = 0$. Then there is a $\sigma < x$ such that

$$\sigma > \sigma_2 \& A^\sigma(w_0) = 0 \& \{r\}^B_{\sigma}w_0 = 0.$$

Since $A$ and $B$ are lightface $\Sigma^1_1$, there is a $\sigma < t_{\delta + 1}$ that satisfies (5). No requirement in any block below block $\delta$, or in block $\delta$ and of higher priority than $r$, receives attention at stage $\sigma$. If $r$ does not need attention at stage $\sigma$, then it was satisfied at an earlier stage by an attempt not yet discarded or injured, and hence never to be discarded or injured. If $r$ does need attention at stage $\sigma$, then it will be satisfied at stage $\sigma$ and remain so forever.  

The proof of Theorem 3.8 was dynamic in nature save for one fine-structure fact:

$$h[t_{\delta} + 1] = L(t_{\delta + 1}).$$
The proof of (6) is a collapsing argument. Nonetheless the proof of Theorem 3.8 can be adjusted so as to avoid (6) (cf. Exercise 3.10). The result is an entirely dynamic, lightface solution to Post’s problem that is uniform with respect to a wide class of structures.

3.9–3.12 Exercises

3.9. Suppose $\alpha^* < \alpha$. Show there exists a lightface, tame $\Sigma^2$ map with range $\alpha$ and domain less than $\alpha$.

3.10. Let $A$ be a $\Sigma_1$ admissible set. Suppose $A$ is effectively wellorderable via a lightface one-one, $A$-recursive map of $A$ onto $\text{ord}(A)$. Show Post’s problem for $A$ has a lightface solution.

3.11. (R. Shore 1974). Suppose $L(\alpha)$ is $\Sigma_n$ admissible, that is, it satisfies $\Sigma_n$ replacement. Find sets $A_i$ ($i \leq 1$) such that $A_i$ is $\Sigma_n$, but is not $\Delta_n^2$, over $\langle L[A_{1-i}, \alpha], \varepsilon, A_{1-i} \rangle$.

3.12. Let $A$ be a $\Sigma_1$ admissible structure of the form $\langle L[\beta, \alpha], \varepsilon, \beta \rangle$. Solve Post’s problem for $A$.

4. $\Sigma_1$ Doing the Work of $\Sigma_2$

Chapter VIII ends on the same note with which it began. Certain $\Sigma_2$ constructions that occur in classical recursion theory can, after some modification, be carried out for every $\Sigma_1$ admissible ordinal. A $\Sigma_2$ construction is one that succeeds by appeal to $\Sigma_2$ replacement. Although the Friedberg-Muchnick (F-M) construction is a $\Sigma_1$ recursion, $\Sigma_2$ replacement is needed to check that every requirement is met. $\sigma(\varepsilon)$, the stage by which the $\varepsilon$-th requirement is met, is a $\Sigma_2$ function of $\varepsilon$; and the proof that the $\varepsilon$-th requirement is met begins with finding a bound on $\{\sigma(c) | c < \varepsilon\}$. Thus the proof that the F-M construction works can be lifted, with no conceptual change, from $L(\omega)$ to every $\Sigma_2$ admissible $L(\alpha)$.

The previous sections of this chapter studied various methods of lifting the F-M construction to every $\Sigma_1$ admissible ordinal. All the methods had in common the idea of projecting $\alpha$ downward to an ordinal with combinatoric properties reminiscent of $\Sigma_2$ replacement. The present section focuses on Lerman’s tame $\Sigma_2$ approach and uses it to lift tame $\Sigma_2$ recursions, in a systematic fashion, from $\omega$ to $\alpha$.

The simplest example of a tame $\Sigma_2$ recursion in classical recursion theory is the construction of a $1$-generic $\Delta_2$ set. After the combinatoric facts about tame $\Sigma_2^2$ recursion are established, they will be applied to obtain a $1$-generic $\Delta_2^2$ set, and finally to prove Simpson’s inversion theorem for the $\alpha$-jump.

4.1 Tame $\Sigma_2$ Recursion. Let $R$ be an $\alpha$-recursively enumerable predicate, and $S$ an $\alpha$-recursive function. A tame $\Sigma_2^2$ recursion is defined by

$$f(\gamma) = \mu y R(f \uparrow \gamma, \gamma, y) \quad \text{if} \quad (E y) R(f \uparrow \gamma, \gamma, y)$$

$$S(f \uparrow \gamma) \quad \text{otherwise.}$$
If $\alpha$ is $\Sigma_2$ admissible, then there exists a unique $\Sigma_2 f$ from $\alpha$ into $\alpha$ satisfying the above recursion equations. If $\alpha$ is merely $\Sigma_1$ admissible, then the recursion may break down at some $\delta$ such that $(f \uparrow \delta) \notin L(\alpha)$. The first such $\delta$ is at least $t\sigma 2p(\alpha)$ according to Lemma 4.2.

A tame $\Sigma_2$ recursion arises out of attempts to meet requirements indexed by ordinals less than $\alpha$. The requirements are simultaneously $\alpha$-recursively enumerable sets. A typical member of a requirement is an ordered pair of disjoint $\alpha$-finite sets. $f(\gamma)$ might be the least extension of $f \uparrow \gamma$ that satisfies requirement $\gamma$. Since the desired extension may not exist, $f$ is $\Sigma_2$ in character rather than $\Sigma_1$.

4.2 Lemma. The equations for a tame $\Sigma_2$ recursion define a unique tame $\Sigma^+_2 f$ from $t\sigma 2p(\alpha)$ into $\alpha$.

Proof. The idea is to construe $f$ as the limit of an $\alpha$-recursive sequence $f_\sigma$. $f(\gamma)$ is approximated at the beginning of stage $\sigma$ by $f_{< \sigma}(\gamma)$, and at the end of stage $\sigma$ by $f_\sigma(\gamma)$. Convergence of $f_\sigma$ to $f$ is assured by a system of priorities. At stage $\sigma$ the best guess for $f(\gamma_1)$ may be inconsistent with the best guess for $f(\gamma_2)$. If $\gamma_1 < \gamma_2$ then $\gamma_1$ is given preference. Thus a guess made for $f(\gamma_2)$ at stage $\sigma$ may be discarded at a later stage for the sake of a new guess for $f(\gamma_1)$. Recall that

$$\lim_{\tau \to \sigma} k(\tau) = z$$

means $(E \rho)_{\rho < \sigma}(\tau)_{\rho < \sigma} [k(\tau) = z]$.

Define (by induction on $\gamma$):

$$f_{< \sigma}(\gamma) = \begin{cases} f(\gamma) & \text{if the limit exists,} \\ S(f_{< \sigma}(\gamma)) & \text{otherwise.} \end{cases}$$

$\gamma$ needs attention at stage $\sigma$ if:

$$L(\sigma) \vdash (Ey) R(f_{< \sigma}(\gamma)) \cap \mu \gamma \leq \mu \gamma \vdash (f_{< \sigma}(\gamma)) \cap \mu \gamma \vdash \mu \gamma \leq \mu \gamma \vdash L(\sigma) \vdash R(f_{< \sigma}(\gamma)) \cap \mu \gamma \vdash \mu \gamma$$

(1)

Let $\gamma^\sigma$ be the least $\gamma$ that needs attention at stage $\sigma$, and $y_0$ the least $y$ alluded to in (1) when $\gamma = \gamma^\sigma$. Define

$$f_{< \sigma}(\gamma) = \begin{cases} f_{< \sigma}(\gamma) & \text{if } \gamma < \gamma^\sigma \\ y_0 & \text{if } \gamma = \gamma^\sigma \end{cases}$$

$$S(f_{< \sigma}(\gamma)) = \begin{cases} S(f_{< \sigma}(\gamma)) & \text{if } \gamma > \gamma^\sigma \end{cases}$$

Let $I_\gamma$ be $\{ \sigma \mid \gamma \geq \gamma^\sigma \}$. It need only be shown that $I_\gamma$ is $\alpha$-finite for all $\gamma < t\sigma 2p(\alpha)$. It then follows from the definition of $f_\sigma$ that $f_{< \sigma}$ converges tamely to some $f$ with domain $t\sigma 2p(\alpha)$. Then $f$, by induction on $\gamma$, is a solution of the tame $\Sigma_2^+_2$ recursion equation.

Fix $\gamma$ to see $I_\gamma$ is $\alpha$-finite. The argument splits into two cases, as it did in Lerman’s tame $\Sigma_1$ approach to Post’s problem in subsection 3.1.
196 VIII. Priority Arguments

Case 1: \( \tau \sigma_2 p(\alpha) \leq g_c(\alpha) \leq \alpha \). Let \( \kappa \) be a regular \( \alpha \)-cardinal such that \( \gamma < \kappa \). Assume \( I_\gamma(x < \gamma) \) is \( \alpha \)-finite and of \( \alpha \)-cardinality less than \( \kappa \). Then the combinatoric lemma (2.3.VII) implies \( \cup \{ I_x | x < \gamma \} \) is \( \alpha \)-finite and of \( \alpha \)-cardinality less than \( \kappa \). It follows from the manner in which \( I_\gamma - \cup \{ I_x | x < \gamma \} \) is interlaced with \( \cup \{ I_x | x < \gamma \} \) that \( I_\gamma \) is \( \alpha \)-finite with \( \alpha \)-cardinality less than \( \kappa \). The interlacing effect is similar to the one discussed in the proof of Lemma 1.3 with one change. Finite sequences of elements of \( I_\gamma - \cup \{ I_x | x < \gamma \} \) are separated by elements of \( \cup \{ I_x | x < \gamma \} \). The change is caused by the insistence that \( f(\gamma) \) be the least \( y \) that satisfies \( R \).

Case 2: \( \tau \sigma_2 p(\alpha) = g_c(\alpha) \cdot \sigma_2 c_f(\alpha) \). Hence

\[
\gamma = g_c(\alpha) \cdot \delta + w
\]

for some \( \delta < \sigma_2 c_f(\alpha) \) and \( w < g_c(\alpha) \). Assume

\[
J_\delta = \cup \{ I_x | g_c(\alpha) \cdot \delta \leq x < g_c(\alpha) \cdot (\delta + 1) \}
\]

is \( \alpha \)-finite for each \( \delta < \delta \). Then \( J_\delta \) is a \( \Sigma_2^+ \) function of \( x \leq \delta \), and so \( \cup \{ J_\delta | z < \delta \} \) is \( \alpha \)-finite, since \( \delta < \sigma_2 c_f(\alpha) \). Let its supremum be \( \sigma_2 \). The argument of Case 1 shows, for each \( x \in (g_c(\alpha) \cdot \delta, g_c(\alpha) \cdot (\delta + 1)) \), that \( I_x - \sigma_2 \) is \( \alpha \)-finite and of \( \alpha \)-cardinality less than some regular \( \alpha \)-cardinal. Hence \( I_\gamma \) is \( \alpha \)-finite.

To complete case 2 it must be shown that \( J_\delta \) is \( \alpha \)-finite. Consider the simultaneous \( \alpha \)-recursive enumeration of \( \{ I_x - \sigma_2 | x \in (g_c(\alpha) \cdot \delta, g_c(\alpha) \cdot (\delta + 1)) \} \). It gives rise to a one-one, \( \alpha \)-recursive map of \( \delta - \sigma_2 \) into \( g_c(\alpha) \cdot (\delta + 1) \). When the \( \rho \)-th entry in the enumeration of \( l_x - \sigma_2 \) appears, it is mapped to \( \langle x - g_c(\alpha) \cdot \delta, \rho \rangle \). \( \rho < g_c(\alpha) \) because the \( \alpha \)-cardinality of \( l_x - \sigma_2 \) is less than some regular \( \alpha \)-cardinal. If \( J_\delta - \sigma_2 \) is not \( \alpha \)-finite, then there is one-one, \( \alpha \)-recursive map of \( \alpha \) onto \( J_\delta - \sigma_2 \). But then \( \alpha^* \leq g_c(\alpha) < \tau \sigma_2 p(\alpha) \), an impossibility by Corollary 2.4.VII. \( \Box \)

4.3 Reduced Tame \( \Sigma_2 \) Recursion. The equations for a tame \( \Sigma_2 \) recursion were formulated with the idea that \( f(\gamma) \) would be defined so as to satisfy requirement \( \gamma \) in some construction involving \( \alpha \) requirements. Since a tame \( \Sigma_2 \) recursion may break down long before \( \alpha \) is reached, it is necessary to re-index requirements. Let \( t \) be a tame \( \Sigma_2^+ \) map from \( \tau \sigma_2 p(\alpha) \) onto \( \alpha \). The equations for tame \( \Sigma_2 \) recursion, reduced by \( t \), are

\[
f(\gamma) = \mu y R(f \uparrow \gamma, t(\gamma), y) \quad \text{if} \quad (E y) R(f \uparrow \gamma, t(\gamma), y)
\]

\[
S(f \uparrow y) \quad \text{otherwise}.
\]

The intention now is that \( f(\gamma) \) be defined to satisfy requirement \( t(\gamma) \). Thus there will be time to satisfy all requirements if the recursion does not break down before stage \( \tau \sigma_2 p(\alpha) \). And it does not, according to the next result.

4.4 Theorem. Let \( t \) be a tame \( \Sigma_2^+ \) map from \( \tau \sigma_2 p(\alpha) \) onto \( \alpha \). Then the equations for a tame \( \Sigma_2^+ \) recursion, reduced by \( t \), define a unique function \( f \) from \( \tau \sigma_2 p(\alpha) \) into \( \alpha \). Furthermore \( f \) is tame \( \Sigma_2^+ \).
Proof. Same as that of Lemma 4.2. The presence of $t$ makes very little difference. At stage $\sigma$, $t(\gamma)$ is guessed at by $t(\sigma, \gamma)$, an $\alpha$-recursive function that converges tamely to $t(\gamma)$. Each of the arguments of 4.3 is altered in the same fashion. First wait for $t(\sigma, \gamma)$ to settle down on the appropriate proper initial segment of $t\sigma_2 p(\sigma)$, and then proceed as in 4.2. □

Theorem 4.4 is a precise interpretation of the phrase: "making $\Sigma_1$ do the work of $\Sigma_2$". It will be applied in the proof of Theorem 4.5 to obtain a 1-generic subset of $\alpha$ for every $\Sigma_1$ admissible $\alpha$. More generally, and less precisely, if a $\Sigma_2$ construction is tame, then it can be executed using only $\Sigma_1$ admissibility.

4.5 1-Genericity. A set $A \subseteq \alpha$ is 1-generic if it is generic with respect to certain bounded $\Pi^2_1$ sentences in the sense defined below. ("Bounded" means the universal quantifier is bounded.)

$p, q, r \ldots$ are forcing conditions. A condition $p$ is a pair $(p^+, p^-)$ of disjoint $\alpha$-finite sets. $p \geq q$ ($p$ is extended by $q$) if $p^+ \subseteq q^+$ and $p^- \subseteq q^-$. $A \in p$ ($A$ satisfies $p$) if $p^+ \subseteq A$ and $p^- \subseteq \alpha - A$. Define

$$\{\gamma\}^p(\delta) = x \text{ by } (p^+, p^-, \delta, x) \in R_p.$$ 

$R_p$ is the $\gamma$-th reduction procedure as in subsection 3.2.VII. Let $\uparrow$ mean undefined, and $\downarrow$ defined. Thus

$$\{\gamma\}^p(\delta) \downarrow \text{ if } (\exists x)[\{\gamma\}^p(\delta) = x].$$

Note that $\{\gamma\}^p(\delta) \downarrow$ is an $\alpha$-recursively enumerable relation on $\gamma, p$ and $\delta$.

$A$ is 1-generic if for all $\gamma, \delta, x < \alpha$:

(1) \quad \left[\{\gamma\}^A(\delta) \uparrow\right] \rightarrow (E p)_{A \in p}(q)_{p \geq q}[\{\gamma\}^q(\delta) \uparrow]; \text{ and}

(2) \quad (\delta)_{\delta < x}[\{\gamma\}^A(\delta) \downarrow] \rightarrow (E p)_{A \in p}(\delta)_{\delta < x}[\{\gamma\}^p(\delta) \downarrow].

A set is said to be tame $\Delta^2_2$ if its characteristic function is tame $\Sigma_2$.

4.6 Theorem. There exists a tame $\Delta^2_2, 1$-generic set.

Proof. Let $S$ be an $\alpha$-recursive function such that

$$S(s) = \left( \bigcup_{\delta < \beta} s^+_\delta, \bigcup_{\delta < \beta} s^-_\delta \right),$$

where $s$ is an $\alpha$-finite sequence (of length $\beta$) of forcing conditions. Let $R(s, \gamma, p)$ be the $\alpha$-recursively enumerable relation

$$S(s) \geq p \quad \& \quad \{\gamma\}_0^p((\gamma)_1) \downarrow.$$

According to Theorem 4.4.VII there is a tame $\Sigma^2_2$ function $f$ with domain $t\sigma 2p(x)$ that satisfies

\[ f(\gamma) = \begin{cases} \mu y R(f\uparrow \gamma, t(\gamma), y) & \text{if } (\exists y) R(f\uparrow \gamma, t(\gamma), y) \\ S(f\uparrow \gamma) & \text{otherwise.} \end{cases} \]

$f$ is a sequence of forcing conditions. $f(\delta_1) \geq f(\delta_2)$ when $\delta_1 \leq \delta_2$. Define

\[ A = \cup \{ f(\gamma) \mid \gamma < t\sigma 2p(x) \}. \]

Suppose $\{\gamma\}^A(\delta) \uparrow$ with the intent of checking clause (1) of the definition of 1-genericity. Choose $x$ so that $(t(x))_0 = \gamma$ and $(t(x))_1 = \delta$, where $t$ is as in subsection 4.3. Let $p = S(f\uparrow x)$. Then $A \in p$ and there is no $q \leq p$ such that $\{\gamma\}^q(\delta) \downarrow$. If there were such a $q$, then $f(x)$ would be such a $q$, $A \in f(x)$, and $\{\gamma\}^A(\delta)$ would be defined.

The checking of clause (2) of the definition of 1-genericity is managed by a trick learned from Normann [1975]. It is based on standard manipulations with closed unbounded subsets of regular cardinals. Let $f_\sigma$ be the $\alpha$-recursive function in the proof of Theorem 4.4. Define

\[ A^{<\sigma} = \cup \{ f_\epsilon(\gamma) \mid \gamma < t\sigma 2p(x) \}. \]

Thus $A^{<\sigma}$ is the $\alpha$-recursive approximation to $A$ provided by the proof of Theorem 4.4 at the beginning of stage $\sigma$.

Assume $(\delta)_\beta < x[\{\gamma\}^A(\delta) \downarrow]$. The “closed unbounded sets” trick is needed to show

(1) \[ (\tau)(E\sigma)_{\geq e}(L(\sigma) \vDash (\delta)_\beta < x[\{\gamma\}^{<\sigma}(\delta) \downarrow]). \]

Suppose for the moment that (1) holds. Let $\epsilon$ be an index such that for all $B \subseteq \alpha$,

\[ \{\epsilon\}^B(0) \downarrow \iff (E\sigma)_{\beta < x}(\delta)_\beta < x[\{\gamma\}^B(\delta) \downarrow]. \]

Choose $y$ so that $(t(y))_0 = \epsilon$ and $(t(y))_1 = 0$. Then (1), and the tame $\Sigma_2$ convergence of $f_\epsilon$ to $f$, imply $\{\gamma\}^{f(\epsilon)}(\delta) \downarrow$ for all $\delta < x$. $A \in f(y)$, so clause (2) of the definition of 1-genericity holds.

It remains to prove (1). Let $\rho$ be the $\alpha$-cardinality of $x$, and $m$ an $\alpha$-finite map of $\rho$ onto $x$. $\gamma_m$ is an index such that

\[ \{\gamma_m\}^B(z) \downarrow \iff \{\gamma\}^B(m(z)) \downarrow \]

for all $B \subseteq \alpha$. Hence

\[ (z)_\rho < x[\{\gamma_m\}^B(z) \downarrow] \iff (\delta)_\beta < x[\{\gamma\}^B(\delta) \downarrow]. \]

Thus it suffices to prove (1) when $x$ is an $\alpha$-cardinal. Proving (1) by induction on the $\alpha$-cardinality of $x$ makes it safe to assume $x$ is a regular $\alpha$-cardinal. (Replace $x$ by its cofinality.) According to Lemma 2.5 there are only two cases.
Case 1: \( \sigma \Sigma p(x) \leq \text{gc}(x) \leq x \). By Exercise 4.14 there is a regular \( x \)-cardinal \( \beta \) and an \( x \)-finite sequence \( y_\delta (\delta < x) \) such that

\[
y_\delta < \beta, \ (t(y_\delta))_0 = \gamma \quad \text{and} \quad (t(y_\delta))_1 = \delta.
\]

Define

\[
k(\sigma, \delta) = \mu_{\tau \geq \sigma} [\{\gamma\}^{f_{\sigma}(\delta)}]_x.
\]

\( k(\sigma, \delta) \) is defined for all \( \sigma < \alpha \) and \( \delta < x \), since

\[
\{\gamma\}^{f_{\sigma}(\delta)}(\delta) \simeq \{\gamma\}^4(\delta) \quad \text{is defined.}
\]

Let \( v \) be an \( x \)-finite map of \( \beta \) onto \( x \) such that \( v^{-1}(\delta) \) is an unbounded subset of \( \beta \) for every \( \delta < x \).

Fix \( \tau \) in order to generate a \( \sigma \) that satisfies the matrix of (1). Let

\[
\sigma_0 = \tau, \quad \sigma_{w+1} = k(\sigma_w, v(w)) \quad (w < \beta), \quad \sigma_\lambda = \sup \{\sigma_w | w < \lambda\} \quad (\lambda \text{ a limit } \leq \beta).
\]

Fix \( \delta \) and consider the behavior of \( \{\gamma\}^{f_{\sigma}(\delta)}(\delta) \) as \( \sigma \rightarrow \sigma_\beta \). The choice of \( v \) implies \( \{\gamma\}^{f_{\sigma}(\delta)}(\delta) \) unboundedly often as \( \sigma \rightarrow \sigma_\beta \). Since \( y_\delta < \beta \) and \( \beta \) is \( x \)-regular, \( f_{\sigma}(y_\delta) \) suffers fewer than \( \beta \) changes as \( \sigma \rightarrow \alpha \). (This point is discussed in Case 1 of the proof of Lemma 4.2. The argument there uses Lemma 2.3.VII to show \( f_{\sigma}(\gamma) \) changes less than \( \kappa \) times, where \( \kappa \) is a regular \( x \)-cardinal and \( \gamma < \kappa \).) Consequently \( f_{\sigma}(y_\delta) \) changes less than \( \beta \) times as \( \sigma \rightarrow \sigma_\beta \). Since the cofinality of \( \sigma_\beta \) is \( \beta \), it follows that \( f_{\sigma}(y_\delta) \) is constant for all sufficiently large \( \sigma \) as \( \sigma \rightarrow \sigma_\beta \). That constant value is \( f_{\sigma}(y_\delta) \), and so \( \{\gamma\}^{f_{\sigma}(\delta)}(\delta) \).

Case 2: \( \sigma \Sigma 2p(x) = \text{gc}(x) \cdot \sigma 2cf(x) \). Similar to Case 1. The idea is to repeat the argument of Case 1 inside the block occupied by \( y \). Define \( \sigma_2 \) as in Case 2 of the proof of Lemma 4.2. Then consider only what happens after stage \( \sigma_2 \). For example, \( f_{\sigma}(y_\delta) \) suffers fewer than \( \beta \) changes after stage \( \sigma_2 \) as \( \sigma \rightarrow \alpha \). \( \square \)

4.7 Proposition. If \( A \) is 1-generic, then \( A \) is regular and hyperregular.

Proof. Suppose \( f \leq_{\omega_1} A \) and \( x < \alpha \). By Lemma 5.2.VII it suffices to show \( f \upharpoonright x \) is \( x \)-finite. If \( f \) is \( \{\gamma\}^4 \), then there exists a \( p \) such that

\[
A \in p \quad \text{and} \quad (\delta)_b < x \quad \{\{\gamma\}^p(\delta) \}. \]

Then \( f \upharpoonright x \) is \( \{\gamma\}^p \upharpoonright x \). \( \square \)

4.8 The \( \alpha \)-Jump. Let \( A \subseteq \alpha \). By analogy with the Turing jump of classical recursion theory, the \( \alpha \)-jump of \( A \) should be a universal, \( \alpha \)-recursively-enumerable-in-\( A \) set. To be precise, the \( \alpha \)-jump of \( A \) should be a set \( B \) such that \( B \) is \( \alpha \)-recursively
enumerable in \( A \), and such that \( C \leq \alpha B \) for every \( C \) \( \alpha \)-recursively enumerable in \( A \).

Three candidates for the definition of “\( B \) is \( \alpha \)-recursively enumerable in \( A \)” come to mind.

(i) There is a \( \gamma < \alpha \) such that

\[
B = \{ \delta \mid \{ \gamma \}^A(\delta) \downarrow \}.
\]

(Recall that “\( \downarrow \)” means “defined”.)

(ii) There is a \( \gamma < \alpha \) such that for all \( \delta < \alpha \),

\[
K_\delta \subseteq B \leftrightarrow \{ \gamma \}^A(\delta) \downarrow.
\]

\( \{ K_\delta \mid \delta < \alpha \} \) is the strong enumeration of \( \alpha \)-finite sets, cf. 3.1.VII.

(iii) \( B \) is \( \Sigma^+_1 \). That is, \( B \) is definable over \( \langle L[A,\alpha], \varepsilon, A \rangle \) by means of a \( \Sigma_1 \) formula with parameters in \( L[A,\alpha] \) with \( x \in A \) regarded as a \( \Delta_0 \) formula.

Clearly (ii) \( \rightarrow \) (i) \( \rightarrow \) (iii). (\( x \rightarrow y \) means if \( B \) is enumerable by definition \( x \), then it is enumerable by definition \( y \).) If \( A \) is regular, then (iii) \( \rightarrow \) (i). If \( A \) is regular and hyperregular, then all three notions agree by Lemma 5.2.VII. Hence if \( A \) is \( \alpha \)-recursively enumerable and hyperregular, then all three notions coincide.

There exists an \( \alpha \) and an \( \alpha \)-recursively enumerable \( A \) on which all three differ.

One objection to (iii) is its dependence on non-\( \alpha \)-finite computations. If (iii) holds, then an element is added to \( B \) because some bounded, but not necessarily \( \alpha \)-finite, set of membership statements is satisfied by \( A \). (ii) and (i) are based on \( \alpha \)-finite computations. (ii) has the virtue of symmetry over (i). If (ii) holds, then an \( \alpha \)-finite set of positive facts about \( B \) follows from an \( \alpha \)-finite set of facts about \( A \). Symmetry considerations led to the rejection of \( \leq_{wa} \) in favor of \( \leq_\alpha \). On the other hand there is in general no universal set as defined above in the sense of (ii).

Thus (i) is the preferred choice for the definition of relative \( \alpha \)-recursive enumerability on which to build the definition of \( \alpha \)-jump. Let

\[
A' \text{ be } \{ \langle \gamma, \delta \rangle \mid \{ \gamma \}^A(\delta) \downarrow \}.
\]

Then \( A' \) is \( \alpha \)-recursively enumerable in \( A \) (in the sense of (i)). Suppose \( B \) is \( \alpha \)-r.e. in \( A \). Thus

\[
B = \{ \delta \mid \{ \gamma_0 \}^A(\delta) \downarrow \}.
\]

Then

\[
H \subseteq B \leftrightarrow \{ \gamma_0 \} \times H \subseteq A', \text{ and}
\]

\[
J \subseteq \alpha - B \leftrightarrow \{ \gamma_0 \} \times J \subseteq \alpha - A'.
\]

Hence \( B \leq_\alpha A' \). If \( A_1 \leq_\alpha A_2 \), then \( A_1' \) is \( \alpha \)-recursively enumerable in \( A_2 \), consequently \( \alpha \)-recursive in \( A_2' \). Thus the \( \alpha \)-jump is well defined on \( \alpha \)-degrees. For any \( \alpha \)-degree, \( d \), let \( d' \), the \( \alpha \)-jump of \( d \), be the \( \alpha \)-degree of \( D \), where \( D \) is any set of degree \( d \).
4. Σ \_1 Doing the Work of Σ \_2

Define \( A^{(n+1)} \) to be \((A^{(n)})'\), and \( A^{(0)} \) to be \( A \).

**Warning**: \( \phi' \) has the same \( \alpha \)-degree as some complete \( \Sigma^\alpha_2 \) set, but in general \( \phi^{(2)} \) does not have the same \( \alpha \)-degree as some complete \( \Sigma^2_2 \) set. Thus the familiar connection between \( \phi^{(n)} \) and \( \Sigma_n \) is broken.

**4.9 Theorem.** If \( A \) is regular and hyperregular, then \( A' \) has the same \( \alpha \)-degree as some regular set \( \alpha \)-recursively enumerable in \( A \).

**Proof.** Since \( A \) is regular and hyperregular, the structure \( \langle L[A, \alpha], \varepsilon, A \rangle \) is \( \Sigma \_1 \) admissible (cf. Exercise 5.8.VII). In addition \( L[A, \alpha] \) equals \( L(\alpha) \). The proof of the regular sets theorem (4.2.VII) is entirely dynamic in nature, and consequently can be extended from \( L(\alpha) \) to \( \langle L[A, \alpha], \varepsilon, A \rangle \) without significant change. The extended version states: if \( B \in \Sigma^\alpha_1 \), then there is a \( C \in \Sigma^\alpha_1 \) such that

(i) \( C \) is regular in the sense of \( L[A, \alpha] \), and
(ii) \( C \equiv_{\alpha, A} B \).

(i) means \((C \cap x) \in L[A, \alpha]\) for all \( x \in L[A, \alpha] \). In (ii), \( \equiv_{\alpha, A} \) refers to reduction procedures that are \( \Sigma^\alpha_1 \) sets of 4-tuples \( \langle H, J, \delta, \gamma \rangle \) from \( L[A, \alpha] \). To visualize the extended proof, extend the natural enumeration of \( L(\alpha) \) to one of \( L[A, \alpha] \).

Since \( L[A, \alpha] = L(\alpha) \), it follows that \( C \) is regular, \( C \oplus A \equiv_{\alpha} B \oplus A \), and \( C \) is \( \alpha \)-recursively enumerable in \( A \). If \( B \) is \( A' \), then \( C \oplus A \) is the desired regular set. \( \square \)

The Simpson jump theorem is a lifting of the Friedberg jump theorem of classical recursion theory. The latter states: \( \phi' \leq d \) iff \( c' = d \) for some \( c \).

**4.10 Theorem** (Simpson 1974a). (i) and (ii) are equivalent.

(i) \( \phi' \leq_{\alpha} D \) and \( D \) has the same \( \alpha \)-degree as some regular set.
(ii) \( C' \equiv_{\alpha} D \) for some regular, hyperregular \( C \).

**Proof.** (ii) implies (i) by Theorem 4.9. Now assume \( D \) satisfies (i). \( C \) is constructed from \( D \) so that

\[
(0) \quad (a) \quad C' \leq_{\alpha} \phi', \; D \quad \text{and} \quad (b) \quad D \leq_{\alpha} C, \; \phi'.
\]

The idea is to make \( C \) 1-generic so that (a) holds, and to code \( D \) into \( C \) so that (b) holds. In addition, the 1-genericity of \( C \) implies \( C \) is regular and hyperregular by Proposition 4.7.

Note that 4.7 makes it safe to assume that all forcing conditions are equivalent to initial segments of characteristic functions. If \( p \) and \( q \) are forcing conditions then \( p \prec q \) is the condition that begins with \( p \) and continues with \( q \).

\[
(p \prec q)(x) = \begin{cases} p(x) & \text{if } x < lh(p) \\ q(x - lh(p)) & \text{if } lh(p) \leq x < lh(p) + lh(q). \end{cases}
\]
Assume $D$ is regular. Let $s$ be an $\alpha$-finite sequence of length $\gamma$ of forcing conditions. Let $S(s)$ be as in the proof of Theorem 4.6. Define

$$S_0(s) = S(s) \cap D \uparrow (t(\gamma)),$$

$$R_0(s, \gamma, q) \leftrightarrow S_0(s) \geq q \quad \& \quad \{ (\gamma)_0 \}^q \uparrow ((\gamma)_1) \downarrow.$$ 

Consider the reduced recursion equations

$$(1) \quad f(\gamma) = \mu y R_0(f \uparrow \gamma, t(\gamma), y) \quad \text{if} \quad (\exists y) R_0(f \uparrow \gamma, t(\gamma), y) \quad S_0(f \uparrow \gamma) \quad \text{otherwise}.$$ 

$t$ is a tame $\Sigma_2^\lambda$ map from $t \sigma 2p(\alpha)$ onto $\alpha$.

Fix $z < t \sigma 2p(\alpha)$ in order to study the recursion defined by (1). As the recursion progresses through $\gamma$’s less than $z$, the only information about $D$ that is needed is

$$D_z = D \uparrow \left( \sup_{\gamma < z} t(\gamma) \right).$$

The regularity of $D$ and tameness of $t$ imply $D_z$ $\alpha$-finite. Let $(1)_2$ be the result of replacing $D$ by $D_z$ in (1). By Theorem 4.4, $(1)_2$ has a tame $\Sigma_2^\lambda$ solution $f_2$ from $t \sigma 2p(\alpha)$ into $\alpha$. The $\alpha$-finite function $f_2 \uparrow z$ is a solution of (1) for $\gamma < z$. Hence

$$\cup \{ f_2 \uparrow z | z < t \sigma 2p(\alpha) \}$$

is a solution of (1). In short, (1) does have a solution $f$, necessarily unique, from $t \sigma 2p(\alpha)$ into $\alpha$. Define

$$C = \cup \{ f(\gamma)^+ | \gamma < t \sigma 2p(\alpha) \}.$$ 

Initial segments of the characteristic function of $C$ correspond to initial segments of $f$. The value of $f \uparrow \gamma$ is determined by (1) from an $\alpha$-finite set of facts about $D, t$ and $\phi'$. $\phi'$ is needed to decide when there is a $\gamma$ that satisfies $R_0$. $R_0$ was defined so that

$$\gamma \in C' \iff (\exists y) R_0(f \uparrow \gamma, t(\gamma), y).$$

It follows that $C' \leq_a \phi', D, t$. The $\Sigma_2^\lambda$ tameness of $t$ implies $t \leq_a \phi'$ (cf. Exercise 4.13). Thus (0) (a) is proved.

A simultaneous recursion on $\gamma$ shows $\lambda \gamma \uparrow f \uparrow \gamma$ and $\lambda \gamma \uparrow D \uparrow t(\gamma)$ are weakly $\alpha$-recursive in $C, \phi'$. Suppose $f \uparrow \gamma$ has been computed $\alpha$-finitely from $C, \phi'$. $I(f \uparrow \gamma) \uparrow D \uparrow t(\gamma)$ is an initial segment of $f(\gamma)$, hence an initial segment of $C$. So $D \uparrow b(\gamma)$ can be extracted from $C, f \uparrow \gamma, t(\gamma)$. Then $f(\gamma)$ can be computed $\alpha$-finitely from $f \uparrow \gamma, D \uparrow t(\gamma), \phi'$. Thus (0) (b) is proved. □

A jump theorem (Sacks 1963) of classical recursion theory that relates recursive enumerability and the Turing jump states: $D$ is recursively enumerable in $\phi'$ and
4.11–4.15 Exercises

4.11. Use Shore's blocking method to prove Theorem 4.10.

4.12. Show a 1-generic set cannot be $\alpha$-recursively enumerable.

4.13. Let $A \subseteq \alpha$. Show $A$ is $\alpha$-recursively enumerable in $\phi'$ iff $A$ is $\Sigma^2_2$. Show $A$ is tame $\Delta^2_2$ iff $A \leq \alpha \phi'$ and $A$ is regular.

4.14. Suppose $t \sigma 2p(\alpha) \leq gc(\alpha)$. Find a tame $\Sigma^2_2$ function $t$ from $t \sigma 2p(\alpha)$ onto $\alpha$ with the following properties. For each $\gamma < \alpha$ and each regular $\alpha$-cardinal $x$, there exists a regular $\alpha$-cardinal $\beta$ and an $\alpha$-finite sequence $y_\delta(\delta < x)$ such that

$$y_\delta < \beta, (t(y_\delta))_0 = \gamma \quad \text{and} \quad (t(y_\delta))_1 = \delta.$$ 

4.15. Provide the details of Case 2 of the proof of Theorem 4.6.