4
Noether's Theorem

4.1 Ignorable Coordinates

A central recurring theme in mathematical physics is the connection between symmetries and conservation laws, in particular the connection between the symmetries of Euclidean space under rotation and translation and the conservation laws for linear and angular momentum. The same connections exist in quantum theory and in relativity, where the symmetry of Minkowski space under translations and Lorentz transformations underlies the relativistic interpretation of energy, momentum, and angular momentum. The connection is particularly transparent in the Lagrangian formalism.

As an example, consider the motion of a free particle in spherical polar coordinates. This is governed by the Lagrangian

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2\sin^2\theta). \tag{4.1}$$

Because $\partial L/\partial\varphi = 0$, we can read off from Lagrange's equations that the angular momentum about the z-axis,

$$\frac{\partial L}{\partial\dot{\varphi}} = mr^2\dot{\varphi}\sin^2\theta \tag{4.2}$$

is conserved.

More generally, suppose that we have a system of particles subject to conservative forces and that we have expressed the Lagrangian $L = T - U$ as a

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function of t and of the generalized position and velocity coordinates q_a and v_a, with $a = 1, 2, \ldots, n$. If L is independent of q_a for some a, then

$$p_a = \frac{\partial L}{\partial v_a} \tag{4.3}$$

is constant along the dynamical trajectories in $P \times \mathbb{R}$ because

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v_a}\right) = \frac{\partial L}{\partial q_a} = 0.$$

Definition 4.1

The quantity p_a is called the *generalized momentum* conjugate to q_a. A coordinate q_a such that $\partial L/\partial q_a = 0$ is said to be *ignorable* or *cyclic.*

Note that the condition

$$\frac{\partial L}{\partial q_1} = 0$$

is a property not just of the coordinate q_1, but of the entire coordinate system $q_1, \ldots, q_n$. It is possible, for example, to introduce new coordinates $\tilde{q}_a$ such that $\tilde{q}_1 = q_1$, but

$$\frac{\partial L}{\partial \tilde{q}_1} \neq \frac{\partial L}{\partial q_1}.$$

It can also happen that $\partial L/\partial \tilde{v}_1 \neq \partial L/\partial v_1$, so that the momentum conjugate to $\tilde{q}_1 = q_1$ need not be the same in the two coordinate systems.

For small ε, the quantity εp_1 is the energy needed to change the generalized velocity v_1 to $v_1 + \varepsilon$, keeping the configuration and the other generalized velocities fixed. This is the sense in which p_1 is a 'momentum'. Linear momentum is a measure of how much energy is needed to change linear velocity by a small amount, angular momentum is a measure of how much energy is required to alter angular velocity.

EXERCISES

4.1. A particle of unit mass moves under gravity on a smooth surface given in cylindrical polar coordinates z, r, θ by $z = f(r)$. Show that the motion is governed by the Lagrangian

$$L = \tfrac{1}{2}\dot{r}^2\left(1 + f'(r)^2\right) + \tfrac{1}{2}r^2\dot{\theta}^2 - gf(r).$$

Show that θ is an ignorable coordinate. Write down the conserved conjugate momentum and give its physical interpretation.

4.2 One-Parameter Transformation Groups

In a Lagrangian system with a cyclic coordinate q_a, the equations of motion are unchanged by adding a constant to q_a. So the existence of cyclic coordinates signals the presence of dynamical symmetry, that is, of the invariance of the dynamics under transformations of the extended configuration space. For example, the conservation of $mr^2\dot{\varphi}\sin^2\theta$ in our example arises from the fact that the Lagrangian and the equations of motion are unchanged by adding a constant to φ, so the dynamical behaviour is unchanged if the system is rotated about the z-axis before being set in motion.

In this example, there is nothing special about the z-axis. We could just as well measure the spherical polar coordinates from the x-axis. Then the cyclic coordinate would be the one associated with symmetry under rotations about the x-axis and the corresponding conserved quantity would be angular momentum about the x-axis.

It would be useful to be able to obtain this second conservation law without actually having to make the transformation to a new system of spherical polar coordinates, and, in general, to be able to spot the presence of a cyclic coordinate of one coordinate system when the Lagrangian is written in terms of another set of generalized coordinates and velocities.

The key to this problem is the recognition that conserved quantities should properly be associated with the action of symmetry groups, rather than with coordinate systems. Linear momenta are associated with the group of translations in space, energy with translation in time, and angular momenta with the action of the rotation group. If the Lagrangian is invariant under such an action, then there is a corresponding conserved quantity.

For the moment, we focus on *one-parameter groups of transformations*, where the group in question is the real numbers, under addition. The transformations act on the extended configuration space $C \times \mathbb{R}$ by changing the configuration of the system, possibly in different ways at different times.

Definition 4.2

A *one-parameter group of transformations* of the extended configuration space is a family of maps $\rho_s : C \times \mathbb{R} \to C \times \mathbb{R}$ labelled by $s \in \mathbb{R}$ such that

(i) ρ_0 is the identity

(ii) $\rho_{s'} \circ \rho_s = \rho_{s+s'}$ for all $s, s' \in \mathbb{R}$

(iii) $t \circ \rho_s = t$

Here t is interpreted as the map $C \times \mathbb{R} \to \mathbb{R}$ that sends (q_a, t) to t.

In coordinates, ρ_s is a family of maps of the form

$$\rho_s : (q_1, \ldots, q_n, t) \mapsto \big(x_1(q, t, s), \ldots, x_n(q, t, s), t\big).$$

So for small s, we have

$$(q_1, \ldots, q_n, t) \mapsto (q_1, \ldots, q_n, t) + s(u_1, \ldots, u_n, 0) + O(s^2),$$

where u_a is the value of $\partial x_a/\partial s$ at $s = 0$. The u_as are functions of the q_as and t, but not s. We call the them the *generators* of the group of transformations. They are the components of a *time-dependent vector field* on configuration space, but we leave this geometric interpretation until later. For the moment, we note the following two key properties.

– From the second property of ρ_s, if we ignore terms of order δs^2, then we have

$$x_a(q, t, s + \delta s) = x_a(q, t, s) + \delta s\, u_a\big(x(q, t, s), t\big).$$

Therefore, with the q_as and t held fixed, $x_a(q, t, s)$ is determined by solving

$$\frac{\mathrm{d}x_a}{\mathrm{d}s} = u_a(x_1, \ldots, x_n, t), \tag{4.4}$$

as a system of ordinary differential equations for the x_as as a functions of s, with the initial condition $x_a(q, t, 0) = q_a$. This is the sense in which the u_as generate ρ_s.

– We can express the generators in a different coordinate system by making a coordinate transformation

$$\tilde{q}_a = \tilde{q}_a(q, t), \qquad \tilde{t} = t.$$

For small δs,

$$\tilde{q}_a(q + \delta s\, u, t) = \tilde{q}_a(q, t) + \delta s\, \frac{\partial \tilde{q}_a}{\partial q_b} u_b,$$

where we have ignored second order terms. Therefore in the new coordinates, the generators are

$$\tilde{u}_a = \frac{\partial \tilde{q}_a}{\partial q_b} u_b. \tag{4.5}$$

Example 4.3

A key example is that in which ρ_s is given in a coordinate system by

$$\rho_s : (q_1, \ldots, q_n, t) \mapsto (q_1 + su_1, \ldots, q_n + su_n, t) \tag{4.6}$$

for some constants u_a. Then the group translates the row vector $(q_1, \ldots, q_n)$ along the constant vector $(u_1, \ldots, u_n)$, leaving t unchanged, and the generators are the components u_a. By solving (4.4), we see that converse is also true. If the generators are constant, then the one-parameter group is given by (4.6).

Example 4.4

For a single particle moving in space, the extended configuration space is $\mathbb{R}^3 \times \mathbb{R}$, with coordinates (x, y, z, t). The one-parameter group of translations in the z direction is given by

$$(x, y, z, t) \mapsto (x, y, z + s, t).$$

The generators are $u_1 = u_2 = 0$, $u_3 = 1$.

The one-parameter group of rotations about the z-axis is given by

$$(x, y, z, t) \mapsto (x \cos s - y \sin s, x \sin s + y \cos s, z, t).$$

The generators are $u_1 = -y$, $u_2 = x$, $u_3 = 0$.

When the generators are constants, the transformations are translations of the coordinates, as in Example 4.3. In this case we say that the one-parameter group is a *dynamical symmetry* of a system with Lagrangian L if

$$u_a \frac{\partial L}{\partial q_a} = 0.$$

It then follows from Lagrange's equations that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(u_a \frac{\partial L}{\partial v_a}\right) = u_a \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v_a}\right) = u_a \frac{\partial L}{\partial q_a} = 0.$$

So the quantity

$$p = u_a \frac{\partial L}{\partial v_a}$$

is a constant of the motion. It is called the *momentum conjugate* to the transformation group. If $u_1 = 1$ and the other u_as are zero, then $p = p_1$, so this extends the terminology of 'coordinates' and 'conjugate momenta'.

When we change to a general coordinate system, the generators transform according to (4.5), and they are no longer constant. However, we do have the following.

Lemma 4.5

Let ρ_s be a one-parameter group of transformations with generators u_a in the coordinate system q_a and with generators $\tilde{u}_a$ in the coordinate system $\tilde{q}_a$, and let $L(q, v, t)$ be a function on $P \times \mathbb{R}$. Then

$$u_a \frac{\partial L}{\partial q_a} + \left(\frac{\partial u_a}{\partial q_b} v_b + \frac{\partial u_a}{\partial t}\right) \frac{\partial L}{\partial v_a} = \tilde{u}_a \frac{\partial L}{\partial \tilde{q}_a} + \left(\frac{\partial \tilde{u}_a}{\partial \tilde{q}_b} \tilde{v}_b + \frac{\partial \tilde{u}_a}{\partial t}\right) \frac{\partial L}{\partial \tilde{v}_a},$$

where $\tilde{u}_a$ and u_a are related by (4.5).

Proof

Suppose that γ is a kinematic trajectory in $C \times R$. Then $\gamma_s = \rho_s(\gamma)$ is a family of trajectories, labelled by s. When we put

$$q_a = q_a(t), \qquad v_a = \dot{q}_a(t)$$

along a dynamical trajectory, L becomes a function of t. If we make this substitution for each trajectory in the family, then the result is a function $L(s,t)$.

Now for small s, the trajectory γ_s is given by

$$q_a = q_a(t) + s u_a\big(q(t), t\big) + O(s^2),$$

where $q_a = q_a(t)$ is the equation of $\gamma_0 = \gamma$. Therefore, along γ_s

$$v_a = \dot{q}_a(t) + s\frac{\partial u_a}{\partial q_b}\dot{q}_b(t) + s\frac{\partial u_a}{\partial t} + O(s^2).$$

It follows that at $s = 0$,

$$\begin{aligned}\frac{\partial L}{\partial s} &= \frac{\partial q_a}{\partial s}\frac{\partial L}{\partial q_a} + \frac{\partial v_a}{\partial s}\frac{\partial L}{\partial v_a} \\ &= u_a\frac{\partial L}{\partial q_a} + \left(\frac{\partial u_a}{\partial q_b}v_b + \frac{\partial u_a}{\partial t}\right)\frac{\partial L}{\partial v_a}.\end{aligned}$$

But $\partial L/\partial s$ is independent of the coordinate system in which it is evaluated. So the lemma follows. □

Definition 4.6

The *derivative* of L under the action of a one-parameter group of transformations is

$$u_a\frac{\partial L}{\partial q_a} + \left(\frac{\partial u_a}{\partial q_b}v_b + \frac{\partial u_a}{\partial t}\right)\frac{\partial L}{\partial v_a}$$

on $P \times \mathbb{R}$. It is independent of the coordinate system in which it is evaluated.

The proof of the lemma gives a method for calculating the derivative. Evaluate L along a trajectory in $C \times \mathbb{R}$ and find the change in L when the trajectory is moved by the action of the group. In geometric language, the action of the group determines a vector field on $P \times \mathbb{R}$ and the derivative is the derivative along this vector field.

The lemma itself gives a means for recognizing dynamical symmetry in general coordinates. If $u_a\partial L/\partial q_a = 0$ in a coordinate system in which the u_as are constant, then (4.6) is zero in this coordinate system and hence in every coordinate system. From (4.5), we also have that

$$u_a \frac{\partial L}{\partial v_a} = \tilde{u}_a \frac{\partial q_b}{\partial \tilde{q}_a} \frac{\partial L}{\partial v_b} = \tilde{u}_a \frac{\partial L}{\partial \tilde{v}_a}$$

so we get the same value for $p = u_a \partial L / \partial v_a$ in every coordinate system.

Definition 4.7

Let $\rho_s : C \times \mathbb{R} \to C \times \mathbb{R}$ be a one parameter group of transformations with generators u_a and let L be a Lagrangian function on $P \times \mathbb{R}$. The *momentum conjugate* to ρ_s is the function

$$p = u_a \frac{\partial L}{\partial v_a} \tag{4.7}$$

on $P \times \mathbb{R}$. The group is said to be a *dynamical symmetry* whenever the derivative of L under the action of the group vanishes.

Proposition 4.8 (Noether's Theorem)

If ρ_s is a dynamical symmetry of a system with Lagrangian $L(q, v, t)$, then its conjugate momentum is constant during the motion of the system.

Proof

By differentiating (4.7) with respect to t along the dynamical trajectories in $P \times \mathbb{R}$, we have

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{\mathrm{d}u_a}{\mathrm{d}t} \frac{\partial L}{\partial v_a} + u_a \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial v_a} \right) \\ &= \left(\frac{\partial u_a}{\partial q_b} v_b + \frac{\partial u_a}{\partial t} \right) \frac{\partial L}{\partial v_a} + u_a \frac{\partial L}{\partial q_a} \\ &= 0. \end{aligned} \tag{4.8}$$

□

In fact, every dynamical symmetry is given by Example 4.3 in some coordinate system, at least in a neighbourhood of a point at which not all the u_as vanish. However, the proof above establishes Noether's theorem direct from the definition, without having to use this result.

Example 4.9

Suppose that q_1 is cyclic. Then the one-parameter transformation group

$$\rho_s : (q_1, q_2, \ldots, q_n, t) \mapsto (q_1 + s, q_2, \ldots, q_n, t)$$

is a dynamical symmetry, with generator $u = (1, 0, \ldots, 0)$. Noether's theorem gives the conservation of the conjugate momentum $p_1 = \partial L / \partial v_1$.

Example 4.10

For a free particle of mass m moving without forces, the Lagrangian is

$$L = \tfrac{1}{2} m (v_1^2 + v_2^2 + v_3^2), \tag{4.9}$$

in the coordinate system $q_1 = x$, $q_2 = y$, and $q_3 = z$. For the one-parameter group of rotations about the z-axis in Example 4.4, we have

$$u_1 = -q_2, \quad u_2 = q_1, \quad u_3 = 0. \tag{4.10}$$

This is a dynamical symmetry because

$$u_a \frac{\partial L}{\partial q_a} + \left(\frac{\partial u_a}{\partial q_b} v_b + \frac{\partial u_a}{\partial t} \right) \frac{\partial L}{\partial v_a} = -v_2 \frac{\partial L}{\partial v_1} + v_1 \frac{\partial L}{\partial v_2} = 0.$$

The conjugate conserved momentum is

$$p = u_a \frac{\partial L}{\partial v_a} = -m q_2 v_1 + m q_1 v_3, \tag{4.11}$$

which is the z-component of the angular momentum $m\boldsymbol{r} \wedge \boldsymbol{v}$.

EXERCISES

4.2. Identify the symmetry associated with the conserved quantity in Exercise 4.1.

4.3. Write down the generators of rotations about the x and y axes for the system in Example 4.10 and find the conjugate momenta. Verify that they are the x- and y-components of angular momentum.

4.3 Conservation of Energy

A conserved momentum arises when L is invariant under a group of instantaneous displacements in the state of motion. Another conservation law arises when L is independent of t. If L is such that

$$L(q, v, t+s) = L(q, v, t),$$

then the system behaves in the same way irrespective of the time at which it is set in motion and so there is symmetry under *time translation*.

We leave the general derivation of the corresponding conservation law for later and deal here only with a special case. Suppose that $L = T - U$, where $U = U(q, t)$ is the potential and T, the kinetic energy, is a homogeneous quadratic in the velocities,

$$T = \tfrac{1}{2} T_{ab}(q, t) v_a v_b. \tag{4.12}$$

Then $E = T + U$ is the *total energy* of the system.

Proposition 4.11

Let $L = T - U$, where T is a homogeneous quadratic in the velocities. If

$$\frac{\partial L}{\partial t} = 0,$$

then E is conserved.

Proof

Suppose that $\partial L/\partial t = 0$. Then $\partial U/\partial t = 0$ and $\partial T/\partial t = 0$ because T and U are homogeneous with different degrees in the velocities. Lagrange's equations are

$$\frac{\partial^2 T}{\partial v_a \partial v_b}\dot{v}_b + \frac{\partial^2 T}{\partial v_a \partial q_b} v_b - \frac{\partial T}{\partial q_a} + \frac{\partial U}{\partial q_a} = 0. \tag{4.13}$$

Hence, by multiplying by v_a and summing over a,

$$0 = \frac{\partial T}{\partial v_b}\frac{\mathrm{d}v_b}{\mathrm{d}t} + \frac{\partial T}{\partial q_b} v_b + v_a \frac{\partial U}{\partial q_a} = \frac{\mathrm{d}}{\mathrm{d}t}(T + U). \tag{4.14}$$

Here we have used

$$v_a \frac{\partial^2 T}{\partial v_a \partial v_b} = \frac{\partial T}{\partial v_b} \quad \text{and} \quad v_a \frac{\partial^2 T}{\partial v_a \partial q_b} = 2\frac{\partial T}{\partial q_b} \tag{4.15}$$

which follow from Euler's theorem on homogeneous functions, because $\partial T/\partial v_b$ is homogeneous of degree one in the velocities and $\partial T/\partial q_b$ is homogeneous of degree two in the velocities. □

4.4 Momentum Principles

Noether's theorem states that the momentum conjugate to a symmetry is a constant of the motion. There is a weaker form of symmetry which, while it does not give rise directly to constants of the motion, is still useful in the analysis of general systems. Suppose that a system of particles has kinetic energy $T = T(q, v, t)$ in generalized coordinates.

Definition 4.12

A *kinematic symmetry* is a one-parameter group of transformations

$$\rho_s : C \times \mathbb{R} \to C \times \mathbb{R}$$

such that the derivative of T under the action of the group vanishes. That is,

$$u_a \frac{\partial T}{\partial q_a} + \left(\frac{\partial u_a}{\partial q_b} v_b + \frac{\partial u_a}{\partial t} \right) \frac{\partial T}{\partial v_a} = 0.$$

The momentum conjugate to a kinematic symmetry with generators u_a is defined to be the function $p : P \to \mathbb{R}$, where

$$p = u_a \frac{\partial T}{\partial v_a}. \tag{4.16}$$

This does not involve the dynamics of the system because the definition does not involve the forces acting on the particles.

If ρ_s is a kinematic symmetry, then the same calculation as in the proof of Noether's theorem, but using now (3.13) rather than Lagrange's equations, shows that

$$\frac{\mathrm{d}p}{\mathrm{d}t} = u_a Q_a \tag{4.17}$$

where p is the conjugate momentum and the Q_as are the generalized forces. In a constrained system subject to workless constraints, they are the q-components of the external forces. Thus a kinematic symmetry gives rise to a *momentum principle* in the form of an equation: rate of change of momentum p equals component of applied force $u_a Q_a$.

We investigate the possibilities first for a single particle with kinetic energy

$$T = \tfrac{1}{2} m \boldsymbol{v} \cdot \boldsymbol{v},$$

where $\boldsymbol{v}$ is the velocity relative an inertial frame. The *linear momentum* of the particle is the vector $\boldsymbol{p} = m\boldsymbol{v}$ and its *angular momentum* about the origin of the inertial frame is the vector

$$\boldsymbol{J} = m\boldsymbol{r} \wedge \boldsymbol{v}$$

where $\boldsymbol{r}$ is the position vector of the particle from the origin of the inertial frame.

Take the generalized coordinates to be the Cartesian coordinates of the particle in the inertial frame. The corresponding coordinates on $P \times \mathbb{R}$ are the three components of $\boldsymbol{r}$, the three components of $\boldsymbol{v}$, and t. We get evolution equations for $\boldsymbol{p}$ and $\boldsymbol{J}$ from (4.17) by considering translations and rotations about the origin of the inertial frame in this coordinate system.

Translations

Suppose that ρ_s moves the particle through distance s in the direction of a unit vector $\boldsymbol{k}$. That is

$$\rho_s : (\boldsymbol{r}, t) \mapsto (\boldsymbol{r} + s\boldsymbol{k}, t).$$

If we take the q_as to be the Cartesian coordinates of the particle, then the generators of the one-parameter group are the components of $\boldsymbol{k}$. Because these are constant and because T depends only on the generalized velocities, T is unchanged by the transformations, so ρ_s is a kinematic symmetry. We have

$$p = m\boldsymbol{k} \cdot \boldsymbol{v} = \boldsymbol{k} \cdot \boldsymbol{p}$$

is the $\boldsymbol{k}$ component of the linear momentum and

$$u_a Q_a = \boldsymbol{k} \cdot \boldsymbol{F},$$

is the $\boldsymbol{k}$-component of the force on the particle. The momentum principle is

$$\dot{p} = \boldsymbol{k} \cdot \boldsymbol{F}, \tag{4.18}$$

which is just the $\boldsymbol{k}$-component of the equation of motion. Because this holds for any $\boldsymbol{k}$, we have the linear momentum equation $\dot{\boldsymbol{p}} = \boldsymbol{F}$.

Rotations about the Origin

In this case, the transformation ρ_s rotates the position vector of the particle through an angle s in about an axis in the direction of a unit vector $\boldsymbol{k}$. To the first order in s, this changes the position and velocity of the particle by

$$(\boldsymbol{r}, \boldsymbol{v}, t) \mapsto (\boldsymbol{r}, \boldsymbol{v}, t) + s(\boldsymbol{k} \wedge \boldsymbol{r}, \boldsymbol{k} \wedge \boldsymbol{v}, t),$$

which leaves T unchanged because $\boldsymbol{v}$ is orthogonal to $\boldsymbol{k} \wedge \boldsymbol{v}$. Therefore ρ_s is a kinematic symmetry. We have

$$p = m(\boldsymbol{k} \wedge \boldsymbol{r}) \cdot \boldsymbol{v} = \boldsymbol{k} \cdot \boldsymbol{J}$$

and

$$u_a Q_a = (\boldsymbol{k} \wedge \boldsymbol{r}) \cdot \boldsymbol{F} = \boldsymbol{k} \cdot (\boldsymbol{r} \wedge \boldsymbol{F})$$

We deduce the *principle of angular momentum*

$$\dot{\boldsymbol{J}} = \boldsymbol{r} \wedge \boldsymbol{F}.$$

The right-hand side is the *moment* of the force $\boldsymbol{F}$ about the origin.

There is an immediate extension to a system of particles with kinetic energy

$$T = \tfrac{1}{2} \sum_\alpha m_\alpha \boldsymbol{v}_\alpha \cdot \boldsymbol{v}_\alpha.$$

The rotations and translations act on $P \times \mathbb{R}$ by acting on the position and velocity of each individual particle, and T is again invariant because each separate term in the sum is invariant. So from the action of the translations we get the *linear momentum* equation

$$\dot{\boldsymbol{p}} = \sum \boldsymbol{F}_\alpha$$

and from the action of the rotations about the origin, we get the *angular momentum equation*

$$\dot{\boldsymbol{J}} = \sum \boldsymbol{r}_\alpha \wedge \boldsymbol{F}_\alpha,$$

where

$$\boldsymbol{p} = \sum m_\alpha \boldsymbol{v}_\alpha \qquad \text{and} \qquad \boldsymbol{J} = \sum m_\alpha \boldsymbol{r}_\alpha \wedge \boldsymbol{v}_\alpha.$$

The forces on a system of particles can usually be split into a sum

$$\boldsymbol{F}_\alpha = \boldsymbol{E}_\alpha + \boldsymbol{I}_\alpha \tag{4.19}$$

where the $\boldsymbol{E}_\alpha$ are *external forces* (gravity and so on) and the $\boldsymbol{I}_\alpha$ are *internal forces* that arise from the mutual interactions between the particles. The only restriction on the internal forces is that they should do no work under small instantaneous displacements that do not alter the relative positions of the particles, that is, under rotations and translations of the whole system. For example, the forces between the particles in a rigid body are internal in this sense. In the absence of gravity and other external forces, one cannot extract energy from a rock simply by moving it or rotating it.

Only the external forces contribute to $u_a Q_a$. So we can rewrite the linear and angular momentum equations in the form

$$\dot{\boldsymbol{p}} = \sum \boldsymbol{E}_\alpha \qquad \text{and} \qquad \dot{\boldsymbol{J}} = \sum \boldsymbol{r}_\alpha \wedge \boldsymbol{E}_\alpha.$$

4.5 Relative Angular Momentum

For a system of particles, we can get a further momentum principle by considering rotations about a *moving* point C. If the point has position vector $\boldsymbol{c}(t)$ from the origin of the inertial frame, then the *angular momentum* $\boldsymbol{J}_C$ of a particle *relative to* C is defined by

$$\boldsymbol{J}_C = m(\boldsymbol{r} - \boldsymbol{c}) \wedge (\boldsymbol{v} - \dot{\boldsymbol{c}})$$

where $\boldsymbol{r}$ is the position vector from the origin of the inertial frame, $\boldsymbol{v}$ is the velocity relative to the frame, and the dot is the time derivative relative to the inertial frame. It depends on C, but is independent of the choice of inertial frame.

The relative angular momentum is related to rotations about the moving point. Let ρ_s denote the one-parameter group of transformations of the extended configuration space of a particle given by rotating the vector $\boldsymbol{r} - \boldsymbol{c}$ from C to the particle through an angle s in about an axis in the direction of a fixed unit vector $\boldsymbol{k}$. To the first order in s,

$$\rho_s : (\boldsymbol{r}, t) \mapsto (\boldsymbol{r}, t) + s\big(\boldsymbol{k} \wedge (\boldsymbol{r} - \boldsymbol{c}), t\big).$$

This changes the position and velocity of a moving particle by

$$(\boldsymbol{r}, \boldsymbol{v}, t) \mapsto (\boldsymbol{r}, \boldsymbol{v}, t) + s\big(\boldsymbol{k} \wedge (\boldsymbol{r} - \boldsymbol{c}), \boldsymbol{k} \wedge (\boldsymbol{v} - \dot{\boldsymbol{c}}), t\big).$$

Therefore

$$p = m\big(\boldsymbol{k} \wedge (\boldsymbol{r} - \boldsymbol{c})\big) \cdot \boldsymbol{v} = \boldsymbol{k} \cdot \boldsymbol{J}_C.$$

The 'momentum' conjugate to the family of rotations is the $\boldsymbol{k}$-component of the angular momentum relative to C.

For a system of particles, we similarly have $p = \boldsymbol{k} \cdot \boldsymbol{J}_C$, where $\boldsymbol{J}_C$ is now the *total angular momentum relative to* C, defined by

$$\boldsymbol{J}_C = \sum m_\alpha (\boldsymbol{r}_\alpha - \boldsymbol{c}) \wedge (\boldsymbol{v}_\alpha - \dot{\boldsymbol{c}}).$$

In general, however, ρ_s is not a kinematic symmetry. For a single particle, the infinitesimal change in T under the action of $\hat{\rho}_s$ for small s is

$$sm\big(\boldsymbol{k} \wedge (\boldsymbol{v} - \dot{\boldsymbol{c}})\big) \cdot \boldsymbol{v} = -s(\boldsymbol{k} \wedge \dot{\boldsymbol{c}}) \cdot \boldsymbol{p},$$

where $\boldsymbol{p}$ is the linear momentum. This does not vanish in general. Obvious exceptions arise when C is fixed, which gives nothing new, or when C is moving with the particle, so that $\dot{\boldsymbol{c}} = \boldsymbol{v}$, in which case the momentum equation is trivial.

For a system of particles, however, there is a more interesting case in which C is taken to be the centre of mass, so that

$$m\boldsymbol{c} = \sum_\alpha m_\alpha \boldsymbol{r}_\alpha,$$

where $m = \sum m_\alpha$ is total mass. In this case the infinitesimal change in T under $\hat{\rho}_s$ is again

$$-s(\boldsymbol{k} \wedge \dot{\boldsymbol{c}}) \cdot \boldsymbol{p},$$

where $\boldsymbol{p}$ is now the total linear momentum. However, with C as the centre of mass, $\boldsymbol{p} = m\dot{\boldsymbol{c}}$, so the change in T vanishes and therefore ρ_s is a kinematic symmetry. We conclude that the time-dependence of the angular momentum about the centre of mass is governed by

$$\dot{\boldsymbol{J}}_c = \sum (\boldsymbol{r} - \boldsymbol{c}) \wedge \boldsymbol{E},$$

where the sum is over the points at which the external forces $\boldsymbol{E}$ are applied.

It is important to remember that the equality between rate of change of angular momentum and total moment of external forces holds in general only for either a fixed point in an inertial frame or for the centre of mass.

EXERCISES

4.4. The configuration space for a particle of mass m moving in space is Euclidean space, with Cartesian coordinates x, y, z. Show that the generator of the one-parameter group of rotations about the x-axis is

$$u = (0, -z, y).$$

Hence write down an expression for the x-component of angular momentum in terms of the spherical polar coordinates defined by

$$\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= r \cos\theta. \end{aligned}$$

4.5. An astronaut is floating in empty space at rest relative to an inertial frame with her arms by her side. Explain how it is that by waving her arms and then returning them to their original position, she can rotate her body, but cannot move her centre of mass.

4.6. A system of particles with masses m_α moves in a gravitational field $\boldsymbol{g}$. Show that if $\boldsymbol{F}_\alpha = -m_\alpha \boldsymbol{g}$ where $\boldsymbol{g}$ is constant, then

$$\sum_\alpha \boldsymbol{F}_\alpha = m\boldsymbol{g} \quad \text{and} \quad \sum_\alpha \boldsymbol{r}_\alpha \wedge \boldsymbol{F}_\alpha = m\boldsymbol{c} \wedge \boldsymbol{g},$$

where $\boldsymbol{c}$ is the position vector of the centre of mass and m is the total mass. Deduce that the effect of a uniform gravitational field on the total linear momentum and angular momentum is the same as that of a single force $m\boldsymbol{g}$ acting through the centre of mass. Show by counter-example that this is not true for a non-uniform field.

4.7. Let $\rho_s : C \times \mathbb{R} \to C \times \mathbb{R}$ be a dynamical symmetry of a system with Lagrangian L, with generators u_a. Show that under the action of ρ_s for small s, the change in $\partial L/\partial v_a$ along a kinematic trajectory is

$$\delta\left(\frac{\partial L}{\partial v_a}\right) = s u_b \frac{\partial^2 L}{\partial v_a \partial q_b} + s w_b \frac{\partial^2 L}{\partial v_a \partial v_b}$$

to the first order in s, where

$$w_a = \frac{\partial u_a}{\partial q_b} v_b + \frac{\partial u_a}{\partial t}.$$

Deduce that

$$\delta\left(\frac{\partial L}{\partial v_a}\right) = -s \frac{\partial u_b}{\partial q_a} \frac{\partial L}{\partial v_b}$$

and

$$\delta\left(\frac{\partial L}{\partial q_a}\right) = -s \frac{\partial u_b}{\partial q_a} \frac{\partial L}{\partial q_b} - s \frac{\partial w_b}{\partial q_a} \frac{\partial L}{\partial v_b}.$$

Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial u_b}{\partial q_a}\right) = \frac{\partial w_b}{\partial q_a}.$$

Hence show that if $q_a = q_a(t)$ is a solution of Lagrange's equations, then so is $q_a = q_a(t) + s u_a\big(q(t), t\big)$, to the first order in s. Deduce that if ρ_s is a dynamical symmetry, then it maps dynamical trajectories in $C \times \mathbb{R}$ to dynamical trajectories.

4.8. Establish the result of Exercise 4.7 by a variational argument.

4.10. Show that in the system in Exercise 3.7, if U depends only on the distance between the particles, then the three components of total angular momentum about the centre of mass are conserved.